

A direct method of solution for the Fokas-Lenells derivative nonlinear Schrödinger equation: II. Dark soliton solutions

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Abstract

In a previous study (Matsuno Y *J. Phys. A: Math. Theor.* **45** (2012) 23202), we have developed a systematic method for obtaining the bright soliton solutions of the Fokas-Lenells derivative nonlinear Schrödinger equation (FL equation shortly) under vanishing boundary condition. In this paper, we apply the method to the FL equation with non-vanishing boundary condition. In particular, we deal with a more sophisticated problem on the dark soliton solutions with a plane wave boundary condition. We first derive the novel system of bilinear equations which is reduced from the FL equation through a dependent variable transformation and then construct the general dark N -soliton solution of the system, where N is an arbitrary positive integer. In the process, a trilinear equation derived from the system of bilinear equations plays an important role. As a byproduct, this equation gives the dark N -soliton solution of the derivative nonlinear Schrödinger equation on the background of a plane wave. We then investigate the properties of the one-soliton solutions in detail, showing that both the dark and bright solitons appear on the nonzero background which reduce to algebraic solitons in specific limits. Last, we perform the asymptotic analysis of the two- and N -soliton solutions for large time and clarify their structure and dynamics.

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1. Introduction

The Fokas-Lenells derivative nonlinear Schrödinger (NLS) equation (FL equation shortly) is a completely integrable nonlinear partial differential equation (PDE) which has been derived as an integrable generalization of the NLS equation using bi-Hamiltonian methods [1]. In the context of nonlinear optics, the FL equation models the propagation of nonlinear light pulses in monomode optical fibers when certain higher-order nonlinear effects are taken into account [2]. We employ the following equation which can be derived from its original version by a simple change of variables combined with a gauge transformation [2]:

$$u_{xt} = u - 2i|u|^2 u_x. \quad (1.1)$$

Here, $u = u(x, t)$ is a complex-valued function of x and t , and subscripts x and t appended to u denote partial differentiations. The complete integrability of the FL equation has been demonstrated by means of the inverse scattering transform (IST) method [3]. Especially, a Lax pair and a few conservation laws associated with it have been obtained explicitly using the bi-Hamiltonian structure and the multisoliton solutions have been derived by applying the dressing method [4]. Another remarkable feature of the FL equation is that it is the first negative flow of the integrable hierarchy of the derivative NLS equation [2, 5].

In a previous study [6] which is referred to as I hereafter, the two different expressions of the bright N -soliton solution of the FL equation have been obtained by a direct method which does not recourse to the IST and their properties have been explored in detail. Here, we construct the dark N -soliton solution of the FL equation on the background of a plane wave. Explicitly, we consider the boundary condition

$$u \rightarrow \rho \exp \{i(\kappa x - \omega t + \phi^{(\pm)})\}, \quad x \rightarrow \pm\infty, \quad (1.2)$$

where $\rho(> 0)$ and κ are real constants representing the amplitude and wavenumber, respectively, $\phi^{(\pm)}$ are real phase constants and the angular frequency $\omega = \omega(\kappa)$ obeys the dispersion relation $\omega = 1/\kappa + 2\rho^2$. Note that the plane wave given in (1.2) is an exact solution of the FL equation. As will be discussed later, the possible values of κ must be

restricted to assure the existence of the soliton solutions. A similar problem to that posed in this paper has been studied recently and an explicit formula for the dark N -soliton solution have been presented by an ingenious application of the Bäcklund transformation between solutions of the FL equation and the Ablowitz-Ladik hierarchy [7]. Nevertheless, the detailed analysis of the soliton solutions has not been undertaken as yet.

An exact method of solution employed here which is sometimes called the direct method [8] or the bilinear transformation method [9] is a powerful tool for analyzing soliton equations and differs from the method used in [7]. Once the equation under consideration is transformed to a system of bilinear equations, the standard technique in the bilinear formalism is applied to obtain soliton solutions. A novel feature of the bilinearization of the FL equation is that one of the bilinear equations can be replaced by a *trilinear* equation, as already demonstrated in I. The same situation happens in the current dark soliton problem. However, the resulting trilinear equation will be used essentially in the process of performing the proof of the dark N -soliton solution.

This paper is organized as follows. In section 2, we bilinearize the FL equation under the boundary condition (1.2). We then show that one of the resulting bilinear equations can be replaced by a trilinear equation. In section 3, we present the dark N -soliton solution of the bilinear equations. It has a simple structure expressed in terms of certain determinants. Subsequently, we perform the proof of the dark N -soliton solution using an elementary theory of determinants in which Jacobi's identity will play a central role. As already noted, the proof of the trilinear equation turns out to be a core in the analysis. In accordance with the relation between the FL equation and the derivative NLS equation at the level of the Lax representation, we also demonstrate that the dark N -soliton solution obtained here yields the dark N -soliton solution of the derivative NLS equation by replacing simply the time dependence of the solution. As in the case of the defocusing NLS equation subjected to nonvanishing boundary conditions, it is necessary for the existence of dark solitons that the asymptotic state given by (1.2) must be stable. Hence, we perform the linear stability analysis of the plane wave solution (1.2) and provide a criterion for the stability. In section 4, we first investigate the properties of the one-soliton solution in detail. We find that depending on the sign of κ and that of the real part of the

complex amplitude parameter, the solution can be classified into two types, i.e., the dark and bright solitons. The latter soliton may be termed "anti-dark soliton" since the background field is nonzero. However, we use a term "bright soliton" throughout the paper. We demonstrate that regardless the sign of κ , the bright soliton has a limiting profile of algebraic type (or an algebraic bright soliton) whereas an algebraic dark soliton appears only if $\kappa < 0$. We then analyze the asymptotic behavior of the two-soliton solution and derive the explicit formulas for the phase shift in terms of the amplitude parameters of solitons. In particular, we address the interaction between a dark soliton and a bright soliton as well as that of two dark solitons. Last, the similar asymptotic analysis to that of the two-soliton solution is performed for the general dark N -soliton solution. Section 5 is devoted to concluding remarks.

2. Exact method of solution

In this section, we develop a direct method of solution for constructing dark soliton solutions of the FL equation (1.1) under the boundary condition (1.2). In particular, we show that it can be transformed to a system of bilinear equations by introducing the same type of the dependent variable transformation as that employed in I for the bilinearization of the FL equation under vanishing boundary condition. We also demonstrate that this system yields a trilinear equation which will play a crucial role in our analysis.

2.1. Bilinearization

The bilinearization of the FL equation (1.1) is established by the following proposition:

Proposition 2.1. *By means of the dependent variable transformation*

$$u = \rho e^{i(\kappa x - \omega t)} \frac{g}{f}, \quad (2.1)$$

with $\omega = 1/\kappa + 2\rho^2$, equation (1.1) can be decoupled into the following system of bilinear equations for the tau functions f and g

$$D_t f \cdot f^* - i\rho^2 (gg^* - ff^*) = 0, \quad (2.2)$$

$$D_x D_t f \cdot f^* - i\rho^2 D_x g \cdot g^* + i\rho^2 D_x f \cdot f^* + 2\kappa\rho^2 (gg^* - ff^*) = 0, \quad (2.3)$$

$$D_x D_t g \cdot f + i\kappa D_t g \cdot f - i\omega D_x g \cdot f = 0. \quad (2.4)$$

Here, $f = f(x, t)$ and $g = g(x, t)$ are complex-valued functions of x and t , and the asterisk appended to f and g denotes complex conjugate and the bilinear operators D_x and D_t are defined by

$$D_x^m D_t^n f \cdot g = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n f(x, t) g(x', t') \Big|_{x'=x, t'=t}, \quad (2.5)$$

where m and n are nonnegative integers.

Proof. Substituting (2.1) into (1.1) and rewriting the resultant equation in terms of the bilinear operators, equation (1.1) can be rewritten as

$$\begin{aligned} & \frac{1}{f^2} (D_x D_t g \cdot f + i\kappa D_t g \cdot f - i\omega D_x g \cdot f) \\ & - \frac{g}{f^3 f^*} \{ f^* D_x D_t f \cdot f - 2\kappa \rho^2 f^2 f^* - 2i\rho^2 g^* (g_x f - g f_x + i\kappa f g) \} = 0. \end{aligned} \quad (2.6)$$

Inserting the identity

$$f^* D_x D_t f \cdot f = f D_x D_t f \cdot f^* - 2f_x D_t f \cdot f^* + f(D_t f \cdot f^*)_x, \quad (2.7)$$

which can be verified by direct calculation, into the second term on the left-hand side of (2.6), one modifies it in the form

$$\begin{aligned} & \frac{1}{f^2} (D_x D_t g \cdot f + i\kappa D_t g \cdot f - i\omega D_x g \cdot f) \\ & - \frac{g}{f^3 f^*} \left[f \{ D_x D_t f \cdot f^* - i\rho^2 D_x g \cdot g^* + i\rho^2 D_x f \cdot f^* + 2\kappa \rho^2 (g g^* - f f^*) \} \right. \\ & \left. - 2f_x \{ D_t f \cdot f^* - i\rho^2 (g g^* - f f^*) \} + f \{ D_t f \cdot f^* - i\rho^2 (g g^* - f f^*) \}_x \right] = 0. \end{aligned} \quad (2.8)$$

By virtue of equations (2.2)-(2.4), the left-hand side of (2.8) vanishes identically. \square

It follows from (2.1) and (2.2) that

$$|u|^2 = \rho^2 + i \frac{\partial}{\partial t} \ln \frac{f^*}{f}. \quad (2.9)$$

The above formula gives the modulus of u in terms of the tau function f .

2.2. Trilinear equation

Proposition 2.2. *The trilinear equation for f and g*

$$f^* \left\{ g_{xt}f - (f_x - i\kappa f)g_t - i \left(\frac{1}{\kappa} + \rho^2 \right) (g_x f - g f_x) \right\} = f_t^* (g_x f - g f_x + i\kappa f g), \quad (2.10)$$

is a consequence of the bilinear equations (2.2)-(2.4).

Proof. By direct calculation, one can show the following trilinear identity among the tau functions f and g :

$$\begin{aligned} & f^* \left\{ g_{xt}f - (f_x - i\kappa f)g_t - i \left(\frac{1}{\kappa} + \rho^2 \right) (g_x f - g f_x) \right\} - f_t^* (g_x f - g f_x + i\kappa f g) \\ &= f^* (D_x D_t g \cdot f + i\kappa D_t g \cdot f - i\omega D_x g \cdot f) \\ & - \frac{g}{2} \left[\{ D_t f \cdot f^* - i\rho^2 (g g^* - f f^*) \}_x + (D_x D_t f \cdot f^* - i\rho^2 D_x g \cdot g^* + i\rho^2 D_x f \cdot f^* - 2i\kappa D_t f \cdot f^*) \right] \\ & + g_x \{ D_t f \cdot f^* - i\rho^2 (g g^* - f f^*) \}. \end{aligned} \quad (2.11)$$

Replacing a term $2i\kappa D_t f \cdot f^*$ on the right-hand side of (2.11) by (2.2), the right-hand side becomes zero by (2.2)-(2.4). This yields (2.10). \square

In view of proposition 2.2, the proof of the dark N -soliton solution is completed if one can prove any three equations among the three bilinear equations (2.2)-(2.4) and a trilinear (2.10). We will see later in section 3 that the proof of (2.4) is not easy to perform and hence we prove (2.10) instead.

3. Dark N -soliton solution and its proof

In this section, we show that the tau functions f and g representing the dark N -soliton solution admit the compact determinantal expressions. This statement is proved by an elementary calculation using the basic formulas for determinants. We first prove that the proposed dark N -soliton solution solves the bilinear equations (2.2) and (2.3) and then the trilinear equation (2.10) in place of (2.4). The implication of the equation (2.10) will be discussed in conjunction with the dark N -soliton solution of the derivative NLS equation. Last, we perform the linear stability analysis of the plane wave solution (1.2) and provide a criterion for the stability.

3.1. Dark N -soliton solution

The main result in this paper is given by the following theorem:

Theorem 3.1. *The dark N -soliton solution of the system of bilinear equations (2.2)-(2.4) is expressed by the following determinants*

$$f = |D|, \quad (3.1a)$$

$$g = \begin{vmatrix} D & \mathbf{z}^T \\ \frac{1}{\rho^2} \mathbf{z}_t^* & 1 \end{vmatrix} = |D| + \frac{1}{\rho^2} \begin{vmatrix} D & \mathbf{z}^T \\ \mathbf{z}_t^* & 0 \end{vmatrix}. \quad (3.1b)$$

Here, D is an $N \times N$ matrix and \mathbf{z} and \mathbf{z}_t are N -component row vectors defined below and the symbol T denotes the transpose:

$$D = (d_{jk})_{1 \leq j, k \leq N}, \quad d_{jk} = \delta_{jk} + \frac{\kappa - ip_j}{p_j + p_k^*} z_j z_k^*, \quad z_j = \exp \left(p_j x + \frac{\kappa \rho^2}{p_j} t + \frac{1}{p_j + i\kappa} \tau + \zeta_{j0} \right), \quad (3.2a)$$

$$\mathbf{z} = (z_1, z_2, \dots, z_N), \quad \mathbf{z}_t = \left(\frac{\kappa \rho^2 z_1}{p_1}, \frac{\kappa \rho^2 z_2}{p_2}, \dots, \frac{\kappa \rho^2 z_N}{p_N} \right), \quad (3.2b)$$

where p_j are complex parameters satisfying the constraints

$$(p_j + i\kappa)(p_j^* - i\kappa) = \frac{1 + \kappa \rho^2}{\kappa \rho^2} p_j p_j^*, \quad j = 1, 2, \dots, N, \quad (3.2c)$$

ζ_{j0} ($j = 1, 2, \dots, N$) are arbitrary complex parameters, δ_{jk} is kronecker's delta and τ is an auxiliary variable.

The dark N -soliton solution is parameterized by $2N$ complex parameters p_j and ζ_{j0} ($j = 1, 2, \dots, N$). The parameters p_j determine the amplitude and velocity of the solitons whereas the parameters ζ_{j0} determine the phase of the solitons. As opposed to the bright soliton case explored in I, however, the real and imaginary parts of p_j are not independent because of the constraints (3.2c). Actually, it may be parameterized either by the velocity of the j th soliton or by a single angular variable, as will see in section 4. An auxiliary variable τ introduced in (3.2a) will be used conveniently in performing the proof of (2.10). It can be set to zero after all the calculations have been completed.

Remark 3.1. The tau function g given by (3.1b) is represented by the determinant of an $(N + 1) \times (N + 1)$ matrix. It can be rewritten by the determinant of an $N \times N$ matrix.

To show this, we multiply the $(N + 1)$ th column of g by $z_{k,t}^*/\rho^2$ and subtract it from the k th column for $k = 1, 2, \dots, N$ to obtain

$$g = \left| \left(\delta_{jk} - \frac{\kappa + ip_k^*}{p_j + p_k^*} \frac{p_j}{p_k^*} z_j z_k^* \right)_{1 \leq j, k \leq N} \right|. \quad (3.3)$$

Although the tau function from (3.1b) is used in the proof of the dark N -soliton solution, an equivalent form (3.3) will be employed in section 4 to analyze the structure of the solution.

Remark 3.2. The complex parameters p_j subjected to the constraints (3.2c) exist only if the condition $\kappa(1 + \kappa\rho^2) > 0$ is satisfied, as confirmed easily by putting $p_j = a_j + ib_j$ with real a_j and b_j . We will show in section 3.6 that this condition is closely related to the stability of the plane wave solution of the FL equation.

3.2. Notation and basic formulas for determinants

Before entering into the proof of the dark N -soliton solution, we first define the matrices associated with the dark N -soliton solution and then provide some basic formulas for determinants. Although these formulas have been used extensively in I, we reproduce them for convenience.

The following bordered matrices appear frequently in our analysis:

$$D(\mathbf{a}; \mathbf{b}) = \begin{pmatrix} D & \mathbf{b}^T \\ \mathbf{a} & 0 \end{pmatrix}, \quad D(\mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{d}) = \begin{pmatrix} D & \mathbf{c}^T & \mathbf{d}^T \\ \mathbf{a} & 0 & 0 \\ \mathbf{b} & 0 & 0 \end{pmatrix}, \quad (3.4)$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and \mathbf{d} are N component row vectors. Let D_{jk} be the cofactor of the element d_{jk} . The following formulas are well known in the theory of determinants [10]:

$$\frac{\partial}{\partial x} |D| = \sum_{j,k=1}^N \frac{\partial d_{jk}}{\partial x} D_{jk}, \quad (3.5)$$

$$\begin{vmatrix} D & \mathbf{a}^T \\ \mathbf{b} & z \end{vmatrix} = |D|z - \sum_{j,k=1}^N D_{jk} a_j b_k, \quad (3.6)$$

$$|D(\mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{d})||D| = |D(\mathbf{a}; \mathbf{c})||D(\mathbf{b}; \mathbf{d})| - |D(\mathbf{a}; \mathbf{d})||D(\mathbf{b}; \mathbf{c})|. \quad (3.7)$$

The formula (3.5) is the differentiation rule of the determinant and (3.6) is the expansion formula for a bordered determinant with respect to the last row and last column. The formula (3.7) is Jacobi's identity. The proof of lemmas described below is based on the above three formulas as well as a few fundamental properties of determinants.

3.3. Differentiation rules and related formulas

In terms of the notation (3.4), the tau functions f and g can be written as

$$f = |D|, \quad (3.8a)$$

$$g = |D| + \frac{1}{\rho^2} |D(\mathbf{z}_t^*; \mathbf{z})|. \quad (3.8b)$$

The differentiation rules of the tau functions with respect to t and x are given by the following formulas:

Lemma 3.1.

$$f_t = \mathbf{i} |D(\mathbf{z}_t^*; \mathbf{z})| - \frac{1}{\rho^2} |D(\mathbf{z}_t^*; \mathbf{z}_t)|, \quad (3.9)$$

$$f_x = -\kappa |D(\mathbf{z}^*; \mathbf{z})| + \mathbf{i} |D(\mathbf{z}^*; \mathbf{z}_x)|, \quad (3.10)$$

$$\begin{aligned} f_{xt} = & \mathbf{i} \kappa \rho^2 |D(\mathbf{z}^*; \mathbf{z})| - \kappa |D(\mathbf{z}_t^*; \mathbf{z})| - \kappa |D(\mathbf{z}^*; \mathbf{z}_t)| + \mathbf{i} |D(\mathbf{z}_t^*; \mathbf{z}_x)| \\ & - |D(\mathbf{z}^*, \mathbf{z}_t^*; \mathbf{z}_x, \mathbf{z})| + \frac{\kappa}{\rho^2} |D(\mathbf{z}^*, \mathbf{z}_t^*; \mathbf{z}, \mathbf{z}_t)| - \frac{\mathbf{i}}{\rho^2} |D(\mathbf{z}^*, \mathbf{z}_t^*; \mathbf{z}_x, \mathbf{z}_t)|, \end{aligned} \quad (3.11)$$

$$g_t = \mathbf{i} |D(\mathbf{z}_t^*; \mathbf{z})| + \frac{1}{\rho^2} |D(\mathbf{z}_{tt}^*; \mathbf{z})|, \quad (3.12)$$

$$g_x = \mathbf{i} |D(\mathbf{z}_t^*; \mathbf{z})| + \frac{1}{\rho^2} |D(\mathbf{z}_t^*; \mathbf{z}_x)| + \frac{\mathbf{i}}{\rho^2} |D(\mathbf{z}_t^*, \mathbf{z}^*; \mathbf{z}, \mathbf{z}_x)|. \quad (3.13)$$

Proof. We prove (3.9). Applying formula (3.5) to f given by (3.1) with (3.2a), one obtains

$$\begin{aligned} f_t &= \kappa \rho^2 \sum_{j,k=1}^N D_{jk} \frac{\kappa - \mathbf{i} p_j}{p_j p_k^*} z_j z_k^* \\ &= -\mathbf{i} \sum_{j,k=1}^N D_{jk} z_j z_{k,t}^* + \frac{1}{\rho^2} \sum_{j,k=1}^N D_{jk} z_{j,t} z_{k,t}^*, \end{aligned}$$

where in passing to the second line, use has been made of the relation $z_{j,t} = (\kappa\rho^2/p_j)z_j$. Referring to formula (3.6) with $z = 0$ and taking into account the notation (3.4), the above expression is equal to the right-hand side of (3.9). Formulas (3.10)-(3.13) can be proved in the same way if one uses (3.5), (3.6) and the relation $\mathbf{z}_{xt} = \kappa\rho^2\mathbf{z}$ as well as some basic properties of determinants. \square

The complex conjugate expressions of the tau functions f and g and their derivatives are expressed as follows:

Lemma 3.2.

$$f^* = |D| - \mathrm{i}|D(\mathbf{z}^*; \mathbf{z})|, \quad (3.14)$$

$$f_t^* = -\mathrm{i}|D(\mathbf{z}^*; \mathbf{z}_t)| - \frac{1}{\rho^2}|D(\mathbf{z}_t^*; \mathbf{z}_t)| + \frac{\mathrm{i}}{\rho^2}|D(\mathbf{z}_t^*, \mathbf{z}^*; \mathbf{z}_t, \mathbf{z})|, \quad (3.15)$$

$$g^* = |D| - \mathrm{i}|D(\mathbf{z}^*; \mathbf{z})| + \frac{1}{\rho^2}|D(\mathbf{z}^*; \mathbf{z}_t)|. \quad (3.16)$$

Proof. It follows from (3.2a) that $d_{jk}^* = d_{kj} + \mathrm{i}z_j^*z_k$ or in the matrix form, $D^* = D^T + \mathrm{i}(z_j z_k^*)_{1 \leq j, k \leq N}^T$. Since $|D^T| = |D|$, one has

$$f^* = |D + \mathrm{i}(z_j z_k^*)_{1 \leq j, k \leq N}| = \begin{vmatrix} D & \mathbf{z}^T \\ -\mathrm{i}\mathbf{z}^* & 1 \end{vmatrix}.$$

Applying formula (3.6) to the right-hand side, formula (3.14) follows immediately. Formulas (3.15) and (3.16) can be derived in the same way. \square

3.4. Proof of the dark N -soliton solution

3.4.1. Proof of (2.2)

Let

$$P_1 = D_t f \cdot f^* - \mathrm{i}\rho^2(gg^* - ff^*). \quad (3.17)$$

Substituting (3.8), (3.9) and (3.14)-(3.16) into (3.17), most terms are canceled, leaving the following three terms

$$P_1 = \frac{\mathrm{i}}{\rho^2} \left\{ -|D||D(\mathbf{z}_t^*, \mathbf{z}^*; \mathbf{z}_t, \mathbf{z})| + |D(\mathbf{z}^*; \mathbf{z})||D(\mathbf{z}_t^*; \mathbf{z}_t)| - |D(\mathbf{z}_t^*; \mathbf{z})||D(\mathbf{z}^*; \mathbf{z}_t)| \right\}.$$

This expression becomes zero by Jacobi's identity. \square

3.4.2. Proof of (2.3)

Instead of proving (2.3) directly, we differentiate (2.2) by x and add the resultant expression to (2.3) and then prove the equation $P_2 = 0$, where

$$P_2 = f_{xt}f^* - f_x f_t^* - i\rho^2(g_x g^* - f_x f^*) + \kappa\rho^2(gg^* - ff^*). \quad (3.18)$$

Substituting (3.8)-(3.11), (3.13) and (3.14)-(3.16) into (3.18) and rearranging, P_2 reduces to

$$\begin{aligned} P_2 = & \frac{\kappa}{\rho^2} \left\{ |D||D(\mathbf{z}^*, \mathbf{z}_t^*, \mathbf{z}, \mathbf{z}_t)| - |D(\mathbf{z}^*; \mathbf{z})||D(\mathbf{z}_t^*; \mathbf{z}_t)| + |D(\mathbf{z}^*; \mathbf{z}_t)||D(\mathbf{z}_t^*; \mathbf{z})| \right\} \\ & + \frac{i}{\rho^2} \left\{ -|D||D(\mathbf{z}^*, \mathbf{z}_t^*, \mathbf{z}_x, \mathbf{z}_t)| + |D(\mathbf{z}^*; \mathbf{z}_x)||D(\mathbf{z}_t^*; \mathbf{z}_t)| - |D(\mathbf{z}^*; \mathbf{z}_t)||D(\mathbf{z}_t^*; \mathbf{z}_x)| \right\} \\ & + \frac{1}{\rho^2} \left\{ -|D(\mathbf{z}^*; \mathbf{z})||D(\mathbf{z}^*, \mathbf{z}_t^*, \mathbf{z}_x, \mathbf{z}_t)| + |D(\mathbf{z}^*; \mathbf{z}_x)||D(\mathbf{z}_t^*, \mathbf{z}^*; \mathbf{z}_t, \mathbf{z})| - |D(\mathbf{z}^*; \mathbf{z}_t)||D(\mathbf{z}^*, \mathbf{z}_t^*, \mathbf{z}, \mathbf{z}_x)| \right\}. \end{aligned} \quad (3.19)$$

The first and second terms on the right-hand side of (3.19) vanish by virtue of Jacobi's identity. To show that the third term becomes zero as well, we consider the determinantal identity

$$\begin{vmatrix} |D(\mathbf{z}^*; \mathbf{z})| & |D(\mathbf{z}^*; \mathbf{z})| & |D(\mathbf{z}_t^*; \mathbf{z})| \\ |D(\mathbf{z}^*; \mathbf{z}_x)| & |D(\mathbf{z}^*; \mathbf{z}_x)| & |D(\mathbf{z}_t^*; \mathbf{z}_x)| \\ |D(\mathbf{z}^*; \mathbf{z}_t)| & |D(\mathbf{z}^*; \mathbf{z}_t)| & |D(\mathbf{z}_t^*; \mathbf{z}_t)| \end{vmatrix} = 0.$$

It is obvious that this determinant is zero since the first two columns coincide. The above assertion follows immediately by expanding the determinant with respect to the first column and using Jacobi's identity. Consequently, $P_2 = 0$. \square

Before proceeding to the proof of (2.10), we emphasize that the constraints (3.2c) have not been used in the process of the proof of (2.2) and (2.3). On the other hand, we find that the proof of (2.4) depends crucially on the constraints. This is an obstacle which has never been encountered in performing the proof of the bright N -soliton solution (see I). In conclusion, a direct proof of (2.4) still remains open and hence we shall prove the trilinear equation (2.10) instead. It turns out, however that its proof is found to be unfeasible. As we shall now demonstrate, introduction of an auxiliary variable τ in the exponential function (3.2) would resolve this difficulty.

3.4.3. Proof of (2.10)

We first prepare the two lemmas to prove (2.10). The lemma 3.3 below gives a very simple relation between the partial derivatives f_t and f_τ . It is to be noted that the constraints (3.2c) are used only for the proof of this lemma.

Lemma 3.3.

$$f_t = (1 + \kappa\rho^2)f_\tau, \quad (3.20a)$$

$$g_t = (1 + \kappa\rho^2)g_\tau. \quad (3.20b)$$

Proof. Extracting the factor z_j from the j th row and the factor z_k^* from the k th column of the determinant $|D|$, respectively for $j, k = 1, 2, \dots, N$, one can rewrite the tau function f into the form

$$f = \prod_{j=1}^N e^{\zeta_j} \left| \left(e^{-\zeta_j} \delta_{jk} + \frac{\kappa - \mathbf{i}p_j}{p_j + p_k^*} \right)_{1 \leq j, k \leq N} \right|,$$

where

$$\zeta_j = (p_j + p_j^*)x + \kappa\rho^2 \left(\frac{1}{p_j} + \frac{1}{p_j^*} \right) t + \frac{p_j + p_j^*}{(p_j + \mathbf{i}\kappa)(p_j^* - \mathbf{i}\kappa)} \tau + \zeta_{j0} + \zeta_{j0}^*,$$

showing that f can be regarded as a function of ζ_j ($j = 1, 2, \dots, N$). Thus, differentiation of f with respect to t gives

$$f_t = \sum_{j=1}^N \frac{\partial f}{\partial \zeta_j} \frac{\partial \zeta_j}{\partial t} = \kappa\rho^2 \sum_{j=1}^N \left(\frac{1}{p_j} + \frac{1}{p_j^*} \right) \frac{\partial f}{\partial \zeta_j}.$$

Similarly, one has

$$f_\tau = \sum_{j=1}^N \frac{p_j + p_j^*}{(p_j + \mathbf{i}\kappa)(p_j^* - \mathbf{i}\kappa)} \frac{\partial f}{\partial \zeta_j}.$$

The constraints (3.2c) are introduced into the above expression to give

$$f_\tau = \frac{\kappa\rho^2}{1 + \kappa\rho^2} \sum_{j=1}^N \left(\frac{1}{p_j} + \frac{1}{p_j^*} \right) \frac{\partial f}{\partial \zeta_j} = \frac{1}{1 + \kappa\rho^2} f_t.$$

This completes the proof of (3.20a). Repeating the similar procedure, one can show that the relation (3.20b) holds as well. \square

The lemma 3.4 gives the differentiation rules of f and g with respect to τ :

Lemma 3.4.

$$f_\tau = \mathbf{i}|D(\mathbf{z}_\tau^*; \mathbf{z})|, \quad (3.21)$$

$$f_\tau^* = -\mathbf{i}|D(\mathbf{z}^*; \mathbf{z}_\tau)|, \quad (3.22)$$

$$g_\tau = \frac{\mathbf{i}}{\kappa\rho^2}|D(\mathbf{z}_t^*; \mathbf{z})| + \frac{1}{\rho^2}|D(\mathbf{z}_t^*; \mathbf{z}_\tau)|, \quad (3.23)$$

$$\begin{aligned} g_{x\tau} = & \mathbf{i}|D(\mathbf{z}^*; \mathbf{z})| + \frac{1}{\rho^2}|D(\mathbf{z}_t^*; \mathbf{z})| + \kappa|D(\mathbf{z}^*; \mathbf{z}_\tau)| + \frac{\mathbf{i}}{\kappa\rho^2}|D(\mathbf{z}_t^*; \mathbf{z}_x)| - \frac{\mathbf{i}\kappa}{\rho^2}|D(\mathbf{z}_t^*; \mathbf{z}_\tau)| \\ & - \frac{1}{\kappa\rho^2}|D(\mathbf{z}_t^*, \mathbf{z}^*; \mathbf{z}, \mathbf{z}_x)| - \frac{\kappa}{\rho^2}|D(\mathbf{z}_t^*, \mathbf{z}^*; \mathbf{z}_\tau, \mathbf{z})| + \frac{\mathbf{i}}{\rho^2}|D(\mathbf{z}_t^*, \mathbf{z}^*; \mathbf{z}_\tau, \mathbf{z}_x)|. \end{aligned} \quad (3.24)$$

Proof. If one notes the relations

$$\mathbf{z}_{t\tau} = -\frac{\mathbf{i}}{\kappa}\mathbf{z}_t + \mathbf{i}\rho^2\mathbf{z}_\tau, \quad \mathbf{z}_{x\tau} = \mathbf{z} - \mathbf{i}\kappa\mathbf{z}_\tau,$$

which follows from the definition (3.2a) of z_j , the proof can be done straightforwardly along with the same procedure as that used in the proof of lemma 3.1 and lemma 3.2. \square

With lemmas (3.2) and (3.3) at hand, we are now ready for starting the proof of (2.10).

Proof of (2.10). If one replaces the t derivative by the τ derivative in accordance with (3.20), the trilinear equation (2.10) can be rewritten in the form

$$f^*P_3 = f_\tau^*P'_3, \quad (3.25a)$$

with the bilinear forms P_3 and P'_3 defined respectively by

$$P_3 = g_{x\tau}f - (f_x - \mathbf{i}\kappa f)g_\tau - \frac{\mathbf{i}}{\kappa}(g_x f - g f_x), \quad (3.25b)$$

$$P'_3 = g_x f - g f_x + \mathbf{i}\kappa f g. \quad (3.25c)$$

The trilinear equation (3.25) is proved as follows. Substituting (3.8), (3.10), (3.13), (3.23) and (3.24) into (3.25b) and applying Jacobi's identity to terms multiplied by $|D|$, P_3 is simplified considerably. After some elementary calculations, one finds that

$$P_3 = \kappa|D(\mathbf{z}^*; \mathbf{z}_\tau)|\left\{|D| + \frac{1}{\rho^2}|D(\mathbf{z}_t^*; \mathbf{z})| - \frac{\mathbf{i}}{\kappa\rho^2}|D(\mathbf{z}_t^*; \mathbf{z}_x)|\right\}. \quad (3.26a)$$

Performing the similar calculation for P'_3 , one obtains

$$P'_3 = \mathrm{i}\kappa \left\{ |D| - |D(\mathbf{z}^*; \mathbf{z})| \right\} \left\{ |D| + \frac{1}{\rho^2} |D(\mathbf{z}_t^*; \mathbf{z})| - \frac{\mathrm{i}}{\kappa\rho^2} |D(\mathbf{z}_t^*; \mathbf{z}_x)| \right\}. \quad (3.26b)$$

Taking into account the formulas (3.14) and (3.22), the expressions (3.26a) and (3.26b) yield (3.25). The trilinear equation (3.25) coupled with lemma 3.3 now completes the proof of the trilinear equation (2.10). \square

3.5. Dark N -soliton solution of the derivative NLS equation

In accordance with the fact that the FL equation is the first negative flow of the Lax hierarchy of the derivative NLS equation, the spatial part of the Lax pair associated with the former equation coincides with that of the latter equation with an identification $q = u_x$ [2, 5]. This observation enables us to obtain the dark N -soliton solution of the derivative NLS equation

$$\mathrm{i}q_t + q_{xx} + 2\mathrm{i}(|q|^2 q)_x = 0, \quad q = q(x, t), \quad (3.27)$$

under the boundary condition

$$q \rightarrow \rho \exp \left\{ \mathrm{i} (\kappa x - \omega' t + \psi^{(\pm)}) \right\}, \quad x \rightarrow \pm\infty, \quad (3.28)$$

where $\omega' = \kappa^2 + 2\kappa\rho^2$ and $\psi^{(\pm)}$ are real phase constants. In particular, we establish the following proposition:

Proposition 3.1. *The dark N -soliton solution of the derivative NLS equation (3.27) subjected to the boundary condition (3.28) is given in terms of the tau functions f and h by*

$$q = \rho \mathrm{e}^{\mathrm{i}(\kappa x - \omega' t)} \frac{h f^*}{f^2}, \quad (3.29a)$$

with

$$f = |D|, \quad h = |H|. \quad (3.29b)$$

Here, D and H are $N \times N$ matrices defined respectively by

$$D = (d_{jk})_{1 \leq j, k \leq N}, \quad d_{jk} = \delta_{jk} + \frac{\kappa - \mathrm{i}p_j}{p_j + p_k^*} z_j z_k^*, \quad z_j = \exp [p_j x + \{\mathrm{i}p_j^2 - 2(\kappa + \rho^2)p_j\}t + \zeta_{j0}], \quad (3.30a)$$

$$H = (h_{jk})_{1 \leq j, k \leq N}, \quad h_{jk} = \delta_{jk} - \frac{\kappa - ip_j}{p_j + p_k^*} \frac{p_j}{p_k^*} z_j z_k^*, \quad (3.30b)$$

where p_j are complex parameters satisfying the constraints

$$p_j p_j^* = \rho^2 \{ \kappa - i(p_j - p_j^*) \}, \quad j = 1, 2, \dots, N, \quad (3.30c)$$

and ζ_{j0} ($j = 1, 2, \dots, N$) are arbitrary complex parameters.

Proof. The correspondence between q and u_x mentioned above implies that the relation

$$q = u_x = \frac{\partial}{\partial x} \left(\rho e^{i\kappa x} \frac{g}{f} \right) = \rho e^{i\kappa x} \frac{1}{f^2} (g_x f - g f_x + i\kappa f g),$$

holds at $t = 0$. On the other hand, the expression in the parentheses on the right-hand side is just P'_3 defined by (3.25c) and hence it is equal to (3.26b). This fact and (3.14) lead, after applying the formula (3.6), to

$$\begin{aligned} g_x f - g f_x + i\kappa f g &= i\kappa \left\{ |D| - |D(\mathbf{z}^*; \mathbf{z})| \right\} \left\{ |D| + \frac{1}{\rho^2} |D(\mathbf{z}_t^*; \mathbf{z})| - \frac{i}{\kappa \rho^2} |D(\mathbf{z}_t^*; \mathbf{z}_x)| \right\} \\ &= i\kappa f^* \begin{vmatrix} D & \left(z_j - \frac{ip_j}{\kappa} z_j \right)_{1 \leq j \leq N}^T \\ \left(\frac{\kappa}{p_k^*} z_k^* \right)_{1 \leq k \leq N} & 1 \end{vmatrix}. \end{aligned}$$

Multiplying the $(N+1)$ th column of the determinant by $\kappa z_k^*/p_k^*$ and subtracting it from the k th column for $k = 1, 2, \dots, N$, one finds that the above expression becomes $i\kappa f^* h$. Consequently,

$$q = i\kappa \rho e^{i\kappa x} \frac{f^* h}{f^2} \Big|_{t=0}.$$

If one replaces q by iq and ρ by ρ/κ , respectively and introduces the time dependence appropriately, one arrives at (3.29) with (3.30). The constraints (3.30c) follow from (3.2c) by the above replacement of ρ . The complex parameters p_j subjected to the constraints (3.30c) exist only if the condition $\kappa + \rho^2 > 0$ is satisfied. \square

It is instructive to perform the bilinearization of the derivative NLS equation under the boundary condition (3.28). This provides an alternative way to construct the dark N -soliton solution given by proposition 3.1, as we shall see now. To this end, following the procedure used in [11, 12], we introduce the gauge transformation

$$q = v \exp \left[i \int_{-\infty}^x (\rho^2 - |v|^2) dx \right], \quad (3.31a)$$

as well as the dependent variable transformation for v

$$v = \rho e^{i(\kappa x - \omega' t)} \frac{h}{f}. \quad (3.31b)$$

Then, equation (3.27) can be decoupled to the system of bilinear equations for f and h

$$D_x f \cdot f^* - i\rho^2 (hh^* - ff^*) = 0, \quad (3.32)$$

$$D_x^2 f \cdot f^* - i\rho^2 D_x h \cdot h^* + \rho^2 (2\kappa + \rho^2)(hh^* - ff^*) = 0, \quad (3.33)$$

$$iD_t h \cdot f + 2i(\kappa + \rho^2)D_x h \cdot f + D_x^2 h \cdot f = 0. \quad (3.34)$$

In view of (3.32), the modulus of v is given in terms of the tau function f by

$$|v|^2 = \rho^2 + i \frac{\partial}{\partial x} \ln \frac{f^*}{f}, \quad (3.35)$$

which, combined with (3.31), yields the formula (3.29). Note from (3.31a) that $|q|^2 = |v|^2$. It may be checked by direct computation that the tau functions f and h from (3.29b) with (3.30) satisfy the above bilinear equations.

It is important to realize that we can take the limit $\kappa \rightarrow 0$ for the solution (3.29) since the dispersion relation is not singular at $\kappa = 0$. This gives the N -soliton solution of the derivative NLS equation on a constant background which has been studied extensively using various exact methods of solution such as the IST [13-16], Bäcklund transformation [17, 18] and Hirota's direct method [19]. On the other hand, for the dark N -soliton solution given by (2.1), this limiting procedure is not relevant because of the singular nature of the dispersion relation.

Last, we shall briefly describe the properties of the one-soliton solution for the purpose of comparison with those of the one-soliton solution of the FL equation. Introducing the new real parameters a_1 and b_1 by $p_1 = a_1 + ib_1$, the square of the modulus of the one-soliton solution from (3.29) and (3.30) with $N = 1$ can be written in the form

$$|q_1|^2 = \rho^2 - \frac{2a_1^2 \operatorname{sgn} a_1}{\sqrt{a_1^2 + (\kappa + b_1)^2}} \frac{1}{\cosh 2(\theta_1 + \delta_1) + \frac{(\kappa + b_1) \operatorname{sgn} a_1}{\sqrt{a_1^2 + (\kappa + b_1)^2}}}. \quad (3.36a)$$

with

$$\theta_1 = a_1(x + c_1 t) + \theta_{10}, \quad c_1 = 2(b_1 + \kappa + \rho^2), \quad e^{4\delta_1} = \frac{a_1^2 + (\kappa + b_1)^2}{4a_1^2}, \quad (3.36b)$$

where $\text{sgn } a_1$ denotes the sign of a_1 , i.e., $a_1 = 1$ for $a_1 > 0$ and $a_1 = -1$ for $a_1 < 0$, and θ_{10} is a real constant. The constraint (3.30c) then becomes

$$a_1^2 + b_1^2 = \rho^2(2b_1 + \kappa). \quad (3.37)$$

Using (3.36b) and (3.37), the parameters a_1 and b_1 are expressed in terms of the velocity c_1 of the soliton as

$$a_1^2 = \frac{1}{4}(c_{\max} - c_1)(c_1 - c_{\min}), \quad b_1 = \frac{c_1}{2} - \kappa - \rho^2, \quad c_{\min} < c_1 < c_{\max}, \quad (3.38a)$$

where

$$c_{\max} = 2(\kappa + 2\rho^2) + 2\rho\sqrt{\kappa + \rho^2}, \quad c_{\min} = 2(\kappa + 2\rho^2) - 2\rho\sqrt{\kappa + \rho^2}. \quad (3.38b)$$

One must impose the condition $\kappa + \rho^2 > 0$ to assure the existence of soliton solutions. Recall that this condition coincides with a criterion for the stability of the plane wave (3.28) [20]. We see from (3.36) that if $a_1 > 0$, then $|q_1|$ takes the form of a dark soliton whereas if $a_1 < 0$, it becomes a bright soliton on a constant background $u = \rho$.

Let A_d and A_b be the amplitudes of the dark and bright solitons, respectively with respect to the background. The amplitude-velocity relations follow from (3.36) and (3.38). They read

$$A_d = \rho - \left| \sqrt{c_1 - \kappa - 2\rho^2} - \sqrt{\kappa + \rho^2} \right|, \quad (3.39a)$$

$$A_b = \sqrt{c_1 - \kappa - 2\rho^2} + \sqrt{\kappa + \rho^2} - \rho. \quad (3.39b)$$

The detailed analysis for the case $\kappa > 0$ has been undertaken in [21]. To sum up, the solution has been shown to exhibit the spiky modulation of the amplitude and phase. It also has been demonstrated that the bright soliton reduces to an algebraic soliton for both limits $c_1 \rightarrow c_{\max}$ and $c_1 \rightarrow c_{\min}$ whereas the algebraic dark soliton never exists. In the case $\kappa < 0$ which has not been treated in [21], however, a careful inspection of (3.36) and (3.38) reveals that the algebraic bright and dark solitons are produced in the limit $c_1 \rightarrow c_{\max}$ and $c_1 \rightarrow c_{\min}$, respectively. The latter new feature is pointed out here for the first time.

Remark 3.3. Using the result obtained in proposition 3.1, we can construct the dark N -soliton solution of the modified NLS equation

$$\mathrm{i}q_t + q_{xx} + \mu|q|^2q + \mathrm{i}\gamma(|q|^2q)_x = 0, \quad q = q(x, t), \quad (3.40)$$

under the boundary condition

$$q \rightarrow \rho \exp \left\{ \mathrm{i} \left(\kappa x - \omega'' t + \psi^{(\pm)} \right) \right\}, \quad x \rightarrow \pm\infty, \quad (3.41)$$

where $\omega'' = \kappa^2 - \mu\rho^2 + \gamma\kappa\rho^2$ and μ and γ are real constants. To show this, we apply the gauge transformation

$$q = \exp \left[\frac{\mu}{\gamma} \tilde{x} + \left(\frac{\mu}{\gamma} \right)^2 \tilde{t} \right] \tilde{q}, \quad x = \tilde{x} + \frac{2\mu}{\gamma} \tilde{t}, \quad t = \tilde{t}, \quad (3.42)$$

to equation (3.40) and see that it can be recast to the derivative NLS equation $\mathrm{i}\tilde{q}_t + \tilde{q}_{\tilde{x}\tilde{x}} + \gamma(|\tilde{q}|^2\tilde{q})_{\tilde{x}} = 0$, which coincides with equation (3.27) with the identification $\tilde{q} = q$, $\tilde{x} = x$, $\tilde{t} = t$ and $\gamma = 2$. The dark N -soliton solution of the equation (3.40) then takes the form

$$q = \rho \mathrm{e}^{\mathrm{i}(\kappa x - \omega'' t)} \frac{f'^* h'}{f'^2}, \quad (3.43a)$$

where the tau functions f' and h' are given respectively by

$$f' = \left| \left(\delta_{jk} + \frac{\kappa - \frac{\mu}{\gamma} - \mathrm{i}p_j}{p_j + p_k^*} z_j z_k^* \right)_{1 \leq j, k \leq N} \right|, \quad (3.43b)$$

$$h' = \left| \left(\delta_{jk} - \frac{\kappa - \frac{\mu}{\gamma} - \mathrm{i}p_j}{p_j + p_k^*} \frac{p_j}{p_k^*} z_j z_k^* \right)_{1 \leq j, k \leq N} \right|, \quad (3.43c)$$

with

$$z_j = \exp \left[p_j x + \{ \mathrm{i}p_j^2 - (2\kappa + \gamma\rho^2)p_j \} t + \zeta_{j0} \right]. \quad (3.43d)$$

The constraints for p_j become

$$p_j p_j^* = \frac{\gamma\rho^2}{2} \left\{ \kappa - \frac{\mu}{\gamma} - \mathrm{i}(p_j - p_j^*) \right\}, \quad j = 1, 2, \dots, N. \quad (3.44)$$

The complex parameters p_j exist only if the condition $\gamma \left(\kappa - \frac{\mu}{\gamma} + \frac{\gamma\rho^2}{2} \right) > 0$ is satisfied.

The following two special cases are worth remarking. The case $\mu = 0$ and $\gamma = 2$ reduces to the result given by proposition 3.1. On the other hand, in the limit $\gamma \rightarrow 0$ while μ being fixed, we first replace z_j by $\sqrt{\gamma}z_j$ for $j = 1, 2, \dots, N$ and then take the limit, producing the dark N -soliton solution of the NLS equation. Note, in this limit, that the constraints (3.44) reduce to $p_j p_j^* = -\mu\rho^2/2$ and hence the dark soliton solutions exist only if the condition $\mu < 0$ is satisfied.

3.6. Stability of the plane wave

We have considered the dark solitons on the background of a plane wave $\rho e^{i(\kappa x - \omega t)}$ with $\omega = 1/\kappa + 2\rho^2$. It is important to see whether the background field is stable or not against perturbations. If unstable, then dark solitons would not exist, as will be demonstrated in the next section. To this end, we perform the linear stability analysis of the plane wave.

Following the standard procedure, we seek a solution of the form

$$u = (\rho + \Delta\rho) e^{i(\kappa x - \omega t + \Delta\phi)}, \quad (3.45)$$

where $\Delta\rho = \Delta\rho(x, t)$ and $\Delta\phi = \Delta\phi(x, t)$ are small perturbations. Substituting (3.45) into the FL equation (1.1) and linearizing about the plane wave, we obtain the system of linear PDEs for $\Delta\rho$ and $\Delta\phi$

$$\Delta\rho_{xt} + \rho(\omega - 2\rho^2)\Delta\phi_x - \kappa\rho\Delta\phi_t - 4\kappa\rho^2\Delta\rho = 0, \quad (3.46a)$$

$$\rho\Delta\phi_{xt} - (\omega - 2\rho^2)\Delta\rho_x + \kappa\Delta\rho_t = 0. \quad (3.46b)$$

Assume the perturbations of the form $e^{i(\lambda x - \nu t)}$ with λ real and ν possibly complex and substitute them into (3.46) to obtain a homogeneous linear system for $\Delta\rho$ and $\Delta\phi$

$$(\lambda\nu - 4\kappa\rho^2)\Delta\rho + i\{\rho\lambda(\omega - 2\rho^2) + \kappa\rho\nu\}\Delta\phi = 0, \quad (3.47a)$$

$$-i\{(\omega - 2\rho^2)\lambda + \kappa\nu\}\Delta\rho + \rho\lambda\nu\Delta\phi = 0. \quad (3.47b)$$

The nontrivial solution exists if ν satisfies the quadratic equation

$$(\lambda^2 - \kappa^2)\nu^2 - 2(2\kappa\rho^2 + 1)\lambda\nu - \frac{\lambda^2}{\kappa^2} = 0. \quad (3.48)$$

Solving this equation, we obtain

$$\nu = \frac{\lambda}{\lambda^2 - \kappa^2} \left[2\kappa\rho^2 + 1 \pm \frac{1}{\kappa} \sqrt{\lambda^2 + 4\kappa^3(\kappa\rho^2 + 1)\rho^2} \right]. \quad (3.49)$$

Thus, if the condition

$$\kappa(\kappa\rho^2 + 1) > 0, \quad (3.50)$$

is satisfied, then ν becomes real for all values of real λ , implying that the plane wave is neutrally stable. It is evident that this condition always holds for $\kappa > 0$. For negative κ , on the other hand, we put $\kappa = -K$ with $K > 0$ and see that the stability criterion turns out to be as $K\rho^2 > 1$. Last, we remark that a similar stability analysis has been performed recently in conjunction with a plane wave solution of the original version of the FL equation [22, 23].

4. Properties of the soliton solutions

In this section, we detail the properties of the soliton solutions. To this end, we first parametrize the complex parameters p_j and ζ_{j0} by the real quantities a_j, b_j, θ_{j0} and χ_{j0} as

$$p_j = a_j + ib_j, \quad \zeta_{j0} = \theta_{j0} + i\chi_{j0}, \quad j = 1, 2, \dots, N, \quad (4.1)$$

and introduce the new independent variables θ_j and χ_j according to the relations

$$\theta_j = a_j(x + c_j t) + \theta_{j0}, \quad c_j = \frac{\kappa\rho^2}{a_j^2 + b_j^2}, \quad j = 1, 2, \dots, N. \quad (4.2a)$$

$$\chi_j = b_j(x - c_j t) + \chi_{j0}, \quad j = 1, 2, \dots, N. \quad (4.2b)$$

In terms of these variables, the variables z_j defined by (3.2a) are put into the form

$$z_j = e^{\theta_j + i\chi_j}, \quad j = 1, 2, \dots, N, \quad (4.2c)$$

after setting $\tau = 0$. Substituting (4.1) into (3.2c), the constraints for p_j can be rewritten as a quadratic equation for b_j

$$b_j^2 - 2\kappa^2\rho^2 b_j + a_j^2 - \kappa^3\rho^2 = 0, \quad j = 1, 2, \dots, N. \quad (4.3)$$

The solution to this equation is found to be as follows:

$$b_j = (\kappa\rho)^2 \pm \sqrt{\kappa^3\rho^2(1 + \kappa\rho^2) - a_j^2}, \quad j = 1, 2, \dots, N. \quad (4.4)$$

We can see from the above expression that the real b_j ($j = 1, 2, \dots, N$) exist only when the condition $\kappa^3\rho^2(1 + \kappa\rho^2) > 0$ is satisfied. This coincides with the criterion (3.50) for the stability of the plane wave, as discussed in section 3.6. Throughout the analysis, we assume this condition to assure the existence of soliton solutions. It is to be noted from (4.2) and (4.3) that the parameters a_j and b_j are expressed in terms of c_j as

$$a_j^2 = \frac{\kappa^2}{4c_j^2} (c_{\max} - c_j)(c_j - c_{\min}), \quad b_j = \frac{1}{2\kappa c_j} (1 - \kappa^2 c_j), \quad c_{\min} < c_j < c_{\max}, \quad (4.5a)$$

where

$$c_{\max} = \frac{1}{\kappa^2} \left\{ 1 + 2\kappa\rho^2 + 2\sqrt{\kappa\rho^2(1 + \kappa\rho^2)} \right\}, \quad c_{\min} = \frac{1}{\kappa^2} \left\{ 1 + 2\kappa\rho^2 - 2\sqrt{\kappa\rho^2(1 + \kappa\rho^2)} \right\}. \quad (4.5b)$$

The relations (4.5) correspond to (3.38) for those of the one-soliton solution of the derivative NLS equation. Thus, the dark N -soliton solution is characterized by the N velocities c_j ($j = 1, 2, \dots, N$) and the $2N$ real phase constants θ_{j0} and χ_{j0} ($j = 1, 2, \dots, N$), the total number of which is $3N$.

Another parameterization of the solution is possible if one introduces the angular variables γ_j by

$$a_j = \sqrt{\kappa^3\rho^2(1 + \kappa\rho^2)} \sin \gamma_j, \quad (4.6a)$$

$$b_j = (\kappa\rho)^2 + \sqrt{\kappa^3\rho^2(1 + \kappa\rho^2)} \cos \gamma_j, \quad 0 < \gamma_j < 2\pi, \quad \gamma_j \neq \pi, \quad j = 1, 2, \dots, N. \quad (4.6b)$$

In terms of γ_j , p_j from (4.1) can be written in the form

$$p_j = i \left\{ (\kappa\rho)^2 + \sqrt{\kappa^3\rho^2(1 + \kappa\rho^2)} e^{-i\gamma_j} \right\}, \quad j = 1, 2, \dots, N, \quad (4.7)$$

and the velocity c_j of the j th soliton given in (4.2a) is expressed as

$$c_j = \frac{1}{\kappa^2 \{1 + 4\kappa\rho^2(1 + \kappa\rho^2) \sin^2 \gamma_j\}} \left\{ 1 + 2\kappa\rho^2 - 2 \operatorname{sgn} \kappa \sqrt{\kappa\rho^2(1 + \kappa\rho^2)} \cos \gamma_j \right\}. \quad (4.8)$$

It follows from the above parametric representation that p_j lies on the circle of radius $\sqrt{\kappa^3\rho^2(1 + \kappa\rho^2)}$ centered at $i(\kappa\rho)^2$ in the complex plane. Let us first describe the properties of the one- and two-soliton solutions and then address the general N -soliton solution.

4.1. One-soliton solution

The tau functions $f = f_1$ and $g = g_1$ for the one-soliton solution follows from (3.1)-(3.3) with $N = 1$. They read

$$f_1 = 1 + \frac{\kappa - ip_1}{p_1 + p_1^*} z_1 z_1^*, \quad g_1 = 1 - \frac{\kappa + ip_1^*}{p_1 + p_1^*} \frac{p_1}{p_1^*} z_1 z_1^*. \quad (4.9)$$

The one-soliton solution u_1 follows from (2.1) with (4.9), yielding

$$u_1 = \rho e^{i(\kappa x - \omega t)} \frac{1 - \frac{\kappa + b_1 + ia_1}{2a_1} \frac{a_1 + ib_1}{a_1 - ib_1} e^{2\theta_1}}{1 + \frac{\kappa + b_1 - ia_1}{2a_1} e^{2\theta_1}}. \quad (4.10)$$

The above expression can be put into the form

$$u_1 = |u_1| e^{i(\kappa x - \omega t)} \exp \{i(\phi + \phi^{(+)})\}, \quad (4.11)$$

where the square of the modulus of u_1 is represented by

$$|u_1|^2 = \rho^2 - \frac{2a_1^2 c \operatorname{sgn}(\kappa a_1)}{\sqrt{a_1^2 + (\kappa + b_1)^2}} \frac{1}{\cosh 2(\theta_1 + \delta_1) + \frac{(\kappa + b_1) \operatorname{sgn} a_1}{\sqrt{a_1^2 + (\kappa + b_1)^2}}}, \quad c = |c_1|, \quad (4.12a)$$

with

$$\theta_1 = a_1(x + c_1 t) + \theta_{10}, \quad c_1 = \frac{\kappa \rho^2}{a_1^2 + b_1^2}, \quad e^{4\delta_1} = \frac{a_1^2 + (\kappa + b_1)^2}{4a_1^2}, \quad (4.12b)$$

and the tangent of the phase ϕ and $\phi^{(+)}$ being given respectively by

$$\tan \phi = \frac{\{a_1^2 + b_1(\kappa + b_1)\} \cosh 2(\theta_1 + \delta_1) + b_1 \operatorname{sgn} a_1 \sqrt{a_1^2 + (\kappa + b_1)^2}}{\kappa a_1 \sinh 2(\theta_1 + \delta_1)}, \quad (4.13a)$$

$$\tan \phi^{(+)} = \frac{a_1^2 + b_1(\kappa + b_1)}{\kappa a_1}. \quad (4.13b)$$

It can be confirmed by direct substitution that (4.11) indeed satisfies the FL equation. The one-soliton solution (4.10) is a one-parameter family of solutions. The parameterization in terms of a_1 will be employed in classifying the soliton solutions. The parameters c_1 and b_1 are then expressed by a_1 . See (4.2a) and (4.4) whereas the parameters ρ and κ are fixed by the boundary condition (1.2). The relation (4.5) will be used conveniently when considering the generation of algebraic solitons in the limit $|a_1| \rightarrow 0$. The form of $|u_1|$

from (4.12) reveals that If $\kappa a_1 > 0$, then $|u_1|$ takes the form of a dark soliton whereas if $\kappa a_1 < 0$, it becomes a bright soliton on a constant background $u = \rho$. Note from (4.12) that the width of the soliton may be defined by $(2|a_1|)^{-1}$. The net change of the phase caused by the effect of nonlinear modulation is given by (4.13). Roughly speaking, the phase ϕ behaves like a step function as a function of θ_1 . Specifically, a rapid change of the phase occurs in the vicinity of the center position of the soliton ($\theta_1 = -\delta_1$), yielding a phase difference π (or $-\pi$). As a result, the phase of u_1 changes by a quantity $2\phi^{(+)}$ as θ_1 varies from $-\infty$ to $+\infty$, where $\phi^{(+)}$ is given by (4.13b).

Let us classify the one-soliton solutions in accordance with the sign of κ . We consider the two cases, i.e., case 1 ($\kappa > 0, a_1 \leq 0$) and case 2 ($\kappa < 0, a_1 \leq 0$) separately. For each sign of κ , both dark and bright solitons arise, as we shall show now.

4.1.1. Case 1: $\kappa > 0$

In this case, the velocity c_1 of the soliton is positive, as evidenced from (4.12b). Let A_d and A_b be the amplitudes of the dark and bright solitons, respectively with respect to the background. We then find from (4.5) and (4.12) that

$$\begin{aligned} A_d &= \rho - \sqrt{\rho^2 - 2c_1 \left\{ \sqrt{a_1^2 + (\kappa + b_1)^2} - (\kappa + b_1) \right\}} \\ &= \rho - \frac{1}{\sqrt{\kappa}} \left| \kappa\sqrt{c} - \sqrt{1 + \kappa\rho^2} \right|, \quad a_1 > 0, \quad c_1 = c > 0, \end{aligned} \quad (4.14)$$

$$\begin{aligned} A_b &= \sqrt{\rho^2 + 2c_1 \left\{ \sqrt{a_1^2 + (\kappa + b_1)^2} + (\kappa + b_1) \right\}} - \rho \\ &= \frac{1}{\sqrt{\kappa}} \left(\kappa\sqrt{c} + \sqrt{1 + \kappa\rho^2} \right) - \rho, \quad a_1 < 0, \quad c_1 = c > 0, \end{aligned} \quad (4.15)$$

where $c \equiv |c_1|$ lies in the interval $c_{\min} < c < c_{\max}$ with c_{\max} and c_{\min} being given by (4.5b). Note from (4.5a) that $\kappa + b_1 = (1 + \kappa^2 c_1)/(2\kappa c_1) > 0$ for $\kappa > 0$ and $c_1 > 0$. This estimate will be used to judge the existence of algebraic solitons in the limit of infinite width.

Figure 1 plots the dependence of the amplitudes $A = A_d$ and $A = A_b$ on the velocity $c = |c_1|$ for $\rho = 1$ and $\kappa = 2$.

(i) *Dark soliton: $a_1 > 0$*

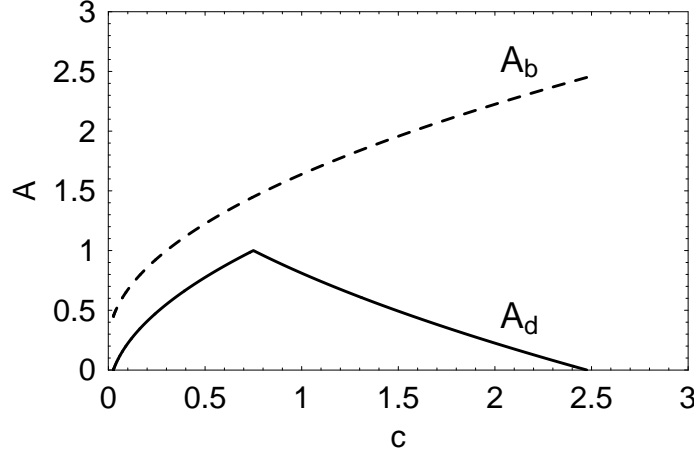


Figure 1. Amplitude-velocity relation for the dark soliton A_d (solid line) and bright soliton A_b (broken line) for $\rho = 1$ and $\kappa = 2$.

As seen from figure 1, the amplitude A_d of the dark soliton becomes an increasing function of the velocity c in the interval $c_{\min} < c \leq c_0$ and a decreasing function in the interval $c_0 < c < c_{\max}$, where c_{\max} ($\gamma_1 = \pi$) and c_{\min} ($\gamma_1 = 0$) are given by (4.5b) and a critical velocity c_0 and the corresponding angle γ_0 by

$$c_0 = \frac{1 + \kappa\rho^2}{\kappa^2}, \quad \text{at } \gamma_1 = \gamma_0 = \cos^{-1} \left[-\frac{(\kappa\rho^2)^{\frac{1}{2}}(3 + 2\kappa\rho^2)}{2(1 + \kappa\rho^2)^{\frac{3}{2}}} \right], \quad (0 < \gamma_0 < \pi). \quad (4.16)$$

In the present numerical example ($\rho = 1, \kappa = 2$), $c_{\min} = 0.025, c_0 = 0.75, c_{\max} = 2.47$. The above observation shows that in the interval $c_0 < c < c_{\max}$, a small dark soliton propagates faster than a large dark soliton. A similar behavior has also been found in I for the bright soliton solutions of the FL equation with zero background.

Figure 2 depicts the profile of $U = |u_1|$ at $t = 0$ for three different values of c , i.e., a: $c = c_0 = 0.75$ ($\gamma_1 = \gamma_0 = 0.90\pi$), b: $c = 0.33$ ($\gamma_1 = 5\pi/6$), c: $c = 0.098$ ($\gamma_1 = 2\pi/3$) with the parameters $\rho = 1, \kappa = 2, \theta_{10} = -\delta_1$ and $\chi_{10} = 0$. When $c = c_0$, the amplitude of the dark soliton attains the maximum value $A_d = \rho$. See figure 2 a. It then turns out that the intensity of the soliton center falls to zero. Such a soliton is well-known in the field of nonlinear optics. It is sometimes called a *black* soliton. For this specific value of c , one

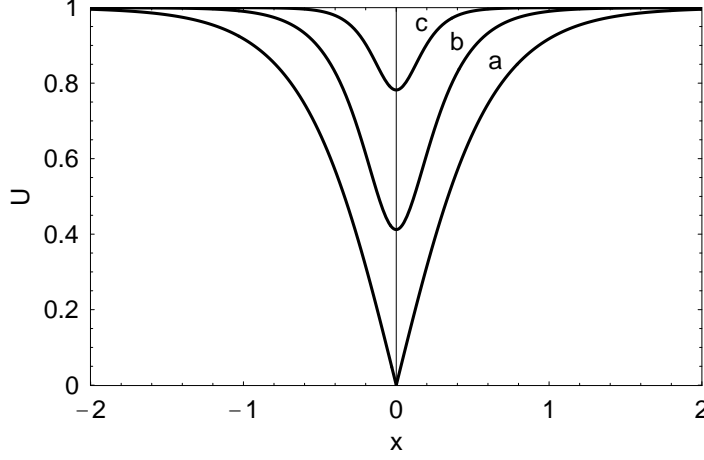


Figure 2. Profile of the amplitude of the dark soliton $U = |u_1|$ at $t = 0$. a: $c = c_0 = 0.75$, b: $c = 0.33$, c: $c = 0.098$. The profile a is a black soliton.

finds from (4.5), (4.12) and (4.13) that

$$a_1 = \frac{\kappa^{\frac{3}{2}} \rho (4 + 3\kappa\rho^2)^{\frac{1}{2}}}{2(1 + \kappa\rho^2)}, \quad b_1 = -\frac{(\kappa\rho)^2}{2(1 + \kappa\rho^2)}, \quad (4.17a)$$

$$e^{4\delta_1} = \frac{\kappa^2}{4a_1^2}, \quad \tan \phi = -\frac{b_1}{a_1} \tanh(\theta_1 + \delta_1), \quad \tan \phi^{(+)} = -\frac{b_1}{a_1}. \quad (4.17b)$$

The profile of $|u_1|^2$ from (4.12) then becomes

$$|u_1|^2 = \rho^2 \left[1 - \frac{4 + 3\kappa\rho^2}{2(1 + \kappa\rho^2)} \frac{1}{\cosh 2(\theta_1 + \delta_1) + \frac{2 + \kappa\rho^2}{2(1 + \kappa\rho^2)}} \right]. \quad (4.18)$$

As confirmed easily from the above expression, the minimum value of $|u_1|$ is zero at $\theta_1 = -\delta_1$. The algebraic dark soliton may be produced from (4.12) by taking the limit $a_1 \rightarrow +0$. However, as already noticed, the value of $\kappa + b_1$ is positive so that $|u_1|$ tends simply to a constant value ρ . Hence, this limiting procedure is irrelevant for the dark soliton solution under consideration, indicating that the algebraic dark soliton does not exist for $\kappa > 0$ and $a_1 > 0$.

Figure 3 shows the profile of $u_R = \text{Re}[u_1]$ at $t = 1$ for the black soliton. The broken line indicates $\pm|u_1|$ (see figure 2 a). One can see that the dark soliton exhibits phase modulations near the center position of the soliton. This peculiar feature is in striking

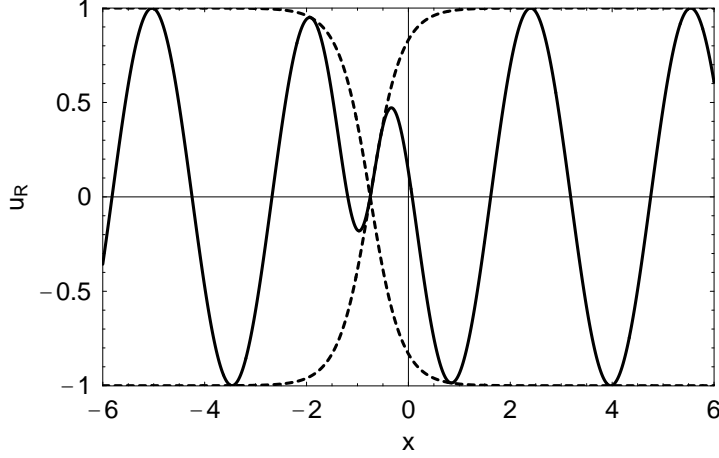


Figure 3. Profile of a black soliton $u_R = \text{Re } u_1$ at $t = 1$.

contrast to the bright soliton solution of the NLS equation for which no phase modulation occurs. A similar behavior has been observed for both dark and bright soliton solutions of the derivative NLS equation with the background of a plane wave [21, 24].

(ii) *Bright soliton:* $a_1 < 0$

Figure 4 depicts the profile of the bright soliton $U = |u_1|$ at $t = 0$ for three different values of c , i.e., a: $c = 2.47(\gamma_1 = 1.001\pi)$, b: $c = 0.73(\gamma_1 = 1.1\pi)$, c: $c = 0.025(\gamma_1 = 1.999\pi)$ with $\rho = 1$ and $\kappa = 2$. The feature of the bright soliton differs substantially from that of the dark soliton. To be specific, the amplitude of the bright soliton always becomes an increasing function of the velocity (see figure 1). It takes the maximum value at $c = c_{\max}(\gamma_1 \rightarrow \pi + 0, a_1 \rightarrow -0)$ and the minimum value at $c = c_{\min}(\gamma_1 \rightarrow 2\pi - 0, a_1 \rightarrow -0)$. At these limiting values of the velocity, the algebraic soliton is produced from the soliton of hyperbolic type. Indeed, if we put $\theta_{10} = a_1 x_0 - \delta_1$ in (4.10) and (4.12) with x_0 being a real constant and then take the limit $a_1 \rightarrow -0$, we find

$$u_1 = \rho e^{i(\kappa x - \omega t)} \frac{x + ct + x_0 - i \frac{2\kappa + b_1}{2b_1(\kappa + b_1)}}{x + ct + x_0 - i \frac{1}{2(\kappa + b_1)}}, \quad (4.19a)$$

$$|u_1|^2 = \rho^2 + \frac{2\kappa c^2}{1 + \kappa^2 c} \frac{1}{(x + ct + x_0)^2 + \left(\frac{\kappa c}{1 + \kappa^2 c}\right)^2}, \quad (4.19b)$$

where $b_1 = (1 - \kappa^2 c)/2\kappa c$ by (4.5a) and $c = c_{\max}$ or c_{\min} . Note from (4.12b) that $b_1^2 = \kappa \rho^2 / c$

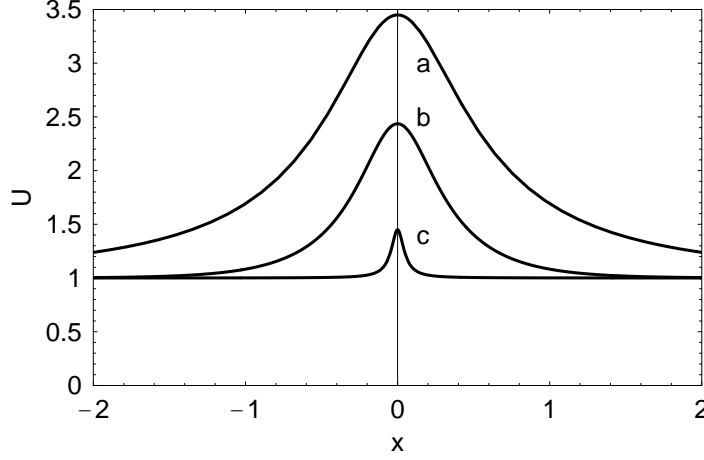


Figure 4. Profile of the amplitude of the bright soliton $U = |u_1|$ at $t = 0$. a: $c = 2.47$, b: $c = 0.73$, c: $c = 0.025$. The profiles a and c are algebraic solitons.

when $a_1 \rightarrow -0$. One can see that the algebraic soliton has no free parameters except a phase constant x_0 since the velocity c is determined by ρ and κ which are fixed by the boundary condition.

To derive (4.19a) from (4.10), we use the following expansion formulas for small a_1 :

$$e^{2\theta_1} = \frac{2|a_1|}{\sqrt{a_1^2 + (\kappa + b_1)^2}} e^{2a_1(x+ct+x_0)} \sim \frac{2|a_1|}{|\kappa + b_1|} \left\{ 1 + 2a_1(x + ct + x_0) + O(a_1^2) \right\}, \quad (4.20a)$$

$$\frac{\kappa + b_1 - ia_1}{2a_1} e^{2\theta_1} \sim \text{sgn } a_1 \text{sgn}(\kappa + b_1) \left[1 + a_1 \left\{ 2(x + ct + x_0) - i \frac{1}{\kappa + b_1} \right\} + O(a_1^2) \right], \quad (4.20b)$$

$$\begin{aligned} & \frac{\kappa + b_1 + ia_1}{2a_1} \frac{a_1 + ib_1}{a_1 - ib_1} e^{2\theta_1} \\ & \sim -\text{sgn } a_1 \text{sgn}(\kappa + b_1) \left[1 + a_1 \left\{ 2(x + ct + x_0) - i \frac{2\kappa + b_1}{b_1(\kappa + b_1)} \right\} + O(a_1^2) \right]. \end{aligned} \quad (4.20c)$$

Because of the inequalities $a_1 < 0$ and $\kappa + b_1 > 0$ in the current problem, one finds that the condition $\text{sgn } a_1 \text{sgn}(\kappa + b_1) = -1$ is satisfied, which yields (4.19a) by taking the limit $a_1 \rightarrow -0$ for (4.10). Actually, under the above condition, the leading-order terms of the denominator and numerator of (4.10) turn out to be of order a_1 . Consequently, the expression (4.10) has a limiting form (4.19a) in the zero limit of a_1 . On the other hand, the expression (4.19b) follows either directly from (4.19a) or from (4.12) by performing the similar limiting procedure.

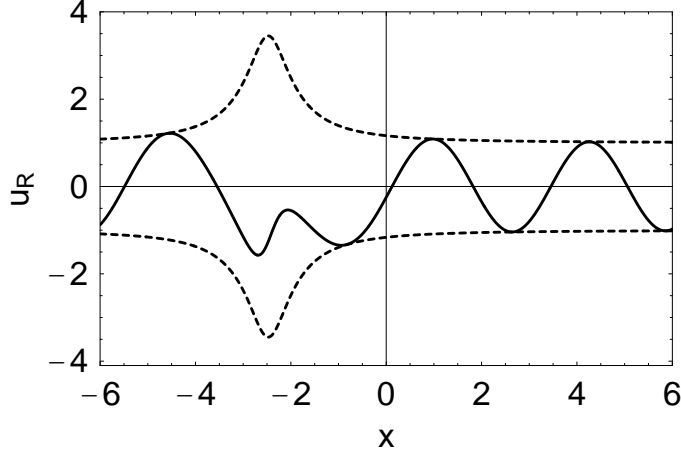


Figure 5. Profile of an algebraic bright soliton $u_R = \text{Re } u_1$ at $t = 1$.

A representative profile of the algebraic bright soliton $U = |u_1|$ at $t = 0$ and the corresponding profile of $u_R = \text{Re } u_1$ at $t = 1$ are shown in figure 4 a and figure 5, respectively.

The novel feature of the bright soliton mentioned above deserves a few comments. First, the amplitude of the bright soliton tends to a finite value when its width tends to infinity, as opposed to the behavior of the dark soliton discussed just before for which the amplitude becomes zero in this limit. Second, the FL equation has an infinite number of conservation laws [3]. Among them, we evaluate the conserved quantity $I = \int_{-\infty}^{\infty} (|u_x|^2 - \kappa^2 \rho^2) dx$ for the one-soliton solution (4.10). This quantity may be termed the energy of the soliton in accordance with the correspondence between the solution u of the FL equation and the solution q of the derivative NLS equation. Using the relation $(|u_x|^2)_t = (|u|^2)_x$ which follows directly from the FL equation, we obtain

$$I = -4 \text{sgn } a_1 \tan^{-1} \left[\frac{1}{|a_1|} \left\{ \sqrt{a_1^2 + (\kappa + b_1)^2} - (\kappa + b_1) \text{sgn } a_1 \right\} \right].$$

We find from this expression that in the limit of infinite width $|a_1| \rightarrow 0$, I becomes zero for the dark soliton ($a_1 > 0$) and tends to a finite value 2π for the bright soliton ($a_1 < 0$). See also an analogous calculation for the bright soliton solution of the derivative NLS equation with zero background [25].

4.1.2. Case 2: $\kappa < 0$

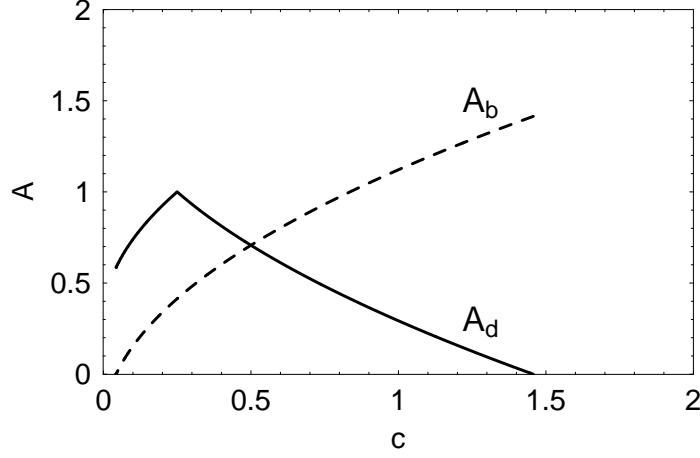


Figure 6. Amplitude-velocity relation for the dark soliton A_d (solid line) and bright soliton A_b (broken line) for $\rho = 1$ and $\kappa = -2$.

For negative κ , the expressions of the amplitude for the dark and bright solitons are given respectively by

$$\begin{aligned}
A_d &= \rho - \sqrt{\rho^2 + 2c_1 \left\{ \sqrt{a_1^2 + (\kappa + b_1)^2} + (\kappa + b_1) \right\}} \\
&= \rho - \frac{1}{\sqrt{K}} \left| K\sqrt{c} - \sqrt{K\rho^2 - 1} \right|, \quad a_1 < 0, \quad c_1 = -c < 0, \quad (4.21)
\end{aligned}$$

$$\begin{aligned}
A_b &= \sqrt{\rho^2 - 2c_1 \left\{ \sqrt{a_1^2 + (\kappa + b_1)^2} - (\kappa + b_1) \right\}} - \rho \\
&= \frac{1}{\sqrt{K}} \left(K\sqrt{c} + \sqrt{K\rho^2 - 1} \right) - \rho, \quad a_1 > 0, \quad c_1 = -c < 0, \quad (4.22)
\end{aligned}$$

where $K = -\kappa$ is a positive wavenumber and the velocity c lies in the interval $c'_{\min} < c < c'_{\max}$ with

$$c'_{\max} = \frac{1}{K^2} \left\{ 2K\rho^2 - 1 + 2\sqrt{K\rho^2(K\rho^2 - 1)} \right\}, \quad c'_{\min} = \frac{1}{K^2} \left\{ 2K\rho^2 - 1 - 2\sqrt{K\rho^2(K\rho^2 - 1)} \right\}. \quad (4.23)$$

Recall that the condition $K\rho^2 - 1 > 0$ must be imposed to assure the existence of the soliton solutions.

Figure 6 plots the dependence of the amplitudes $A = A_d$ and $A = A_b$ on the velocity $c = |c_1|$ for $\rho = 1$ and $\kappa = -2$. When compared with figure 1 for $\kappa > 0$, there appear

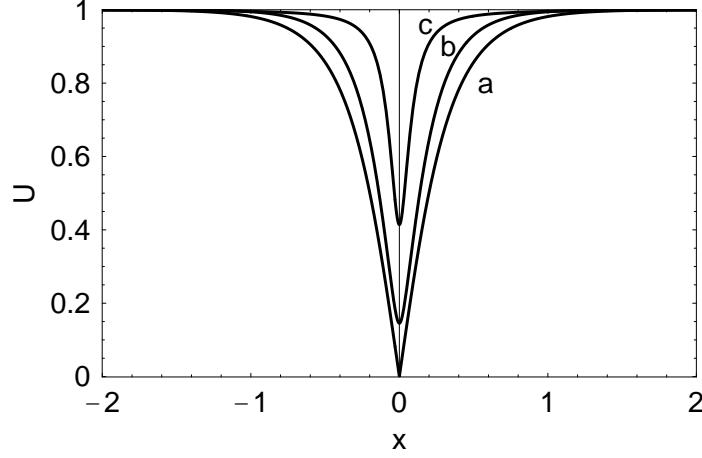


Figure 7. Profile of the amplitude of the dark soliton $U = |u_1|$ at $t = 0$. a: $c = c_0 = 0.25$, b: $c = 0.16$, c: $c = 0.043$. The profile a is a black soliton and the profile c is an algebraic soliton.

several different features for $\kappa < 0$. In particular, the algebraic *dark* soliton would arise in the limit $c \rightarrow c'_{\min}$ since in this limit, the amplitude A_d tends to a finite value. In addition, the algebraic bright soliton exists only in the limit $c \rightarrow c'_{\max}$. We now proceed to the detailed description of the soliton solutions.

(i) *Dark soliton:* $a_1 < 0$

It follows from (4.5) with $\kappa = -K, c_1 = -c$ that $\kappa + b_1 = 1/2Kc - K/2$. Since $c'_{\min} < c < c'_{\max}$ by (4.23), the possible value of $\kappa + b_1$ is restricted by the inequality

$$K \left[K\rho^2 - 1 - \sqrt{K\rho^2(K\rho^2 - 1)} \right] < \kappa + b_1 < K \left[K\rho^2 - 1 + \sqrt{K\rho^2(K\rho^2 - 1)} \right]. \quad (4.24)$$

One can see that the upper limit of $\kappa + b_1$ is attained when $c = c'_{\min}$ and its limiting value is positive by the condition $K\rho^2 > 1$ whereas the lower limit is attained when $c = c'_{\max}$ and is negative. In view of this fact, the algebraic dark soliton would be produced in the limit $c \rightarrow c'_{\min}$ for which $\text{sgn}(\kappa + b_1) > 0$. Actually, taking the limit $a_1 \rightarrow -0$ for the solutions (4.10) and (4.12) and using the expansion formulas (4.20), we find that the hyperbolic soliton reduces to the limiting form

$$u_1 = \rho e^{i(-Kx - \omega t)} \frac{x - ct + x_0 - i \frac{-2K + b_1}{2b_1(-K + b_1)}}{x - ct + x_0 - i \frac{1}{2(-K + b_1)}}, \quad (4.25a)$$

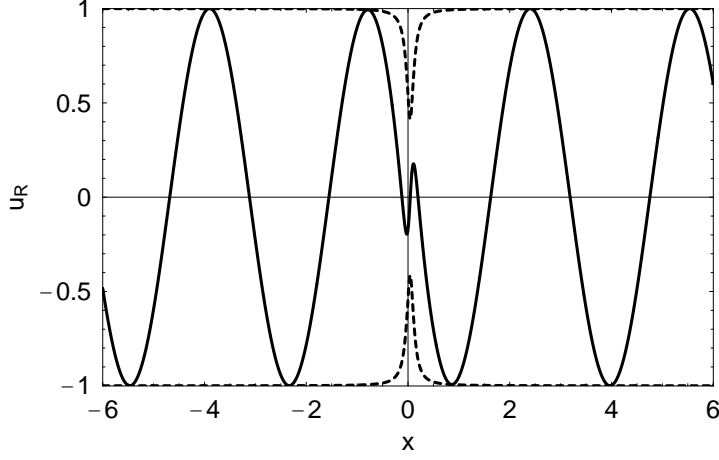


Figure 8. Profile of an algebraic dark soliton $u_R = \text{Re } u_1$ at $t = 1$.

$$|u_1|^2 = \rho^2 - \frac{2Kc^2}{1 - K^2c} \frac{1}{(x - ct + x_0)^2 + \left(\frac{Kc}{1 - K^2c}\right)^2}, \quad (4.25b)$$

where $b_1 = (1 + K^2c)/2Kc$ and $c = c'_{\min}$. Since $1 - Kc'_{\min} > 0$ by virtue of the condition $K\rho^2 > 1$, the expression (4.25b) actually represents an algebraic dark soliton.

The black soliton appears when the velocity c takes a specific value $c = c'_0$, where

$$c'_0 = (K\rho^2 - 1)/K^2 \quad \text{at} \quad \gamma_1 = \gamma'_0 = \cos^{-1} \left[\frac{(K\rho^2)^{\frac{1}{2}}(3 - 2K\rho^2)}{2(K\rho^2 - 1)^{\frac{3}{2}}} \right], \quad (\pi < \gamma'_0 < 2\pi). \quad (4.26)$$

Its profile is represented by

$$|u_1|^2 = \rho^2 \left[1 - \frac{3K\rho^2 - 4}{2(K\rho^2 - 1)} \frac{1}{\cosh 2(\theta_1 + \delta_1) + \frac{K\rho^2 - 2}{2(K\rho^2 - 1)}} \right]. \quad (4.27)$$

It is important to notice that the inequality $c'_{\min} < c'_0 < c'_{\max}$ requires the condition $K\rho^2 > 4/3$ for the wavenumber K . It then turns out that expression (4.27) takes the form of a black soliton.

Figure 7 depicts the profile of $U = |u_1|$ at $t = 0$ for three different values of c , i.e., a: $c = c'_0 = 0.25(\gamma_1 = \gamma'_0 = 5\pi/4)$, b: $c = 0.16(\gamma_1 = 4\pi/3)$, c: $c = 0.043(\gamma_1 = 2\pi)$ with the parameters $\rho = 1, \kappa = -2, \theta_{10} = -\delta_1$ and $\chi_{10} = 0$. In this example, $c'_{\min} = 0.043, c'_0 = 0.25$ and $c'_{\max} = 1.46$ (see figure 6). An algebraic soliton appears at the lower limit of the

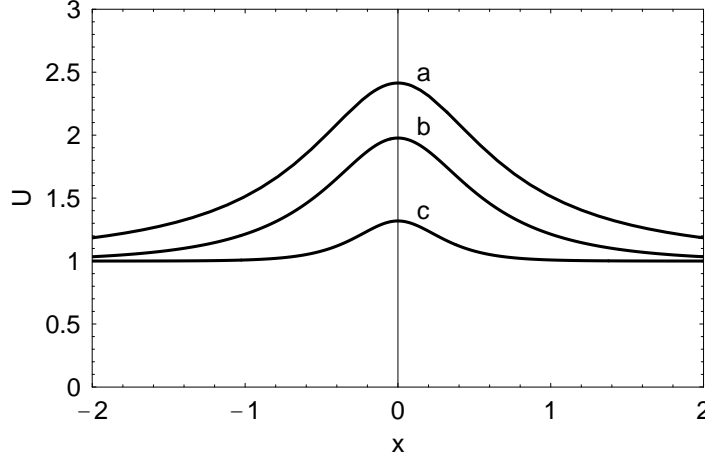


Figure 9. Profile of the amplitude of the bright soliton $U = |u_1|$ at $t = 0$. a: $c = 1.46$, b: $c = 0.81$, c: $c = 0.19$. The profiles a is an algebraic soliton.

velocity, i.e., $c = c'_{\min}$ whereas a black soliton arises at $c = c'_0$. Figure 8 shows the profile of $u_R = \text{Re } u_1$ at $t = 1$ for an algebraic dark soliton.

(ii) *Bright soliton: $a_1 > 0$*

The crucial difference between the case 1 and the case 2 for the bright solitons is observed if one compares figure 6 with figure 1. Notably, the bright soliton with $\kappa < 0$ reduces to an algebraic soliton only at the upper limit of the velocity $c = c'_{\max}$ whereas the bright soliton with $\kappa > 0$ has two critical velocities c_{\max} and c_{\min} for which algebraic solitons are produced. Figure 9 depicts the profile of $U = |u_1|$ at $t = 0$ for three different values of c , i.e., a: $c = 1.46$ ($\gamma_1 = 0.998\pi$), b: $c = 0.73$ ($\gamma_1 = 0.9\pi$), c: $c = 0.025$ ($\gamma_1 = 0.7\pi$) with $\rho = 1$ and $\kappa = -2$. Figure 10 shows the profile $u_R = \text{Re } u_1$ of an algebraic bright soliton at $t = 1$ which corresponds to the profile a in figure 9.

4.1.3. Note on algebraic solitons

We have seen that the algebraic solitons arise from the hyperbolic solitons when certain conditions are satisfied. Here, we summarize the result. The algebraic bright solitons are produced when the conditions $\text{sgn } a_1 \text{sgn}(\kappa + b_1) = -1$ and $\text{sgn}(\kappa a_1) < 0$ are satisfied simultaneously whereas the corresponding conditions for the dark algebraic solitons are given by $\text{sgn } a_1 \text{sgn}(\kappa + b_1) = -1$ and $\text{sgn}(\kappa a_1) > 0$. Thus, for $\kappa > 0$, the conditions

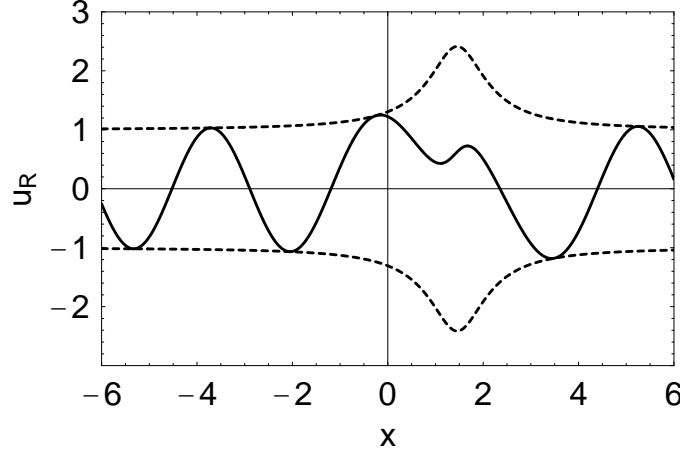


Figure 10. Profile of an algebraic bright soliton $u_R = \text{Re } u_1$ at $t = 1$.

$\text{sgn}(\kappa + b_1) = 1$ and $\text{sgn}(\kappa + b_1) = -1$ are responsible for the generation of the algebraic bright and dark solitons, respectively. Since $\kappa + b_1 > 0$ in this case, only the bright algebraic soliton exists. See figure 1. For $\kappa < 0$, on the other hand, the above conditions turn out to be $\text{sgn}(\kappa + b_1) = -1$ and $\text{sgn}(\kappa + b_1) = 1$, respectively. Under this setting, the limiting value of $\kappa + b_1$ becomes negative for the bright soliton and positive for the dark soliton, respectively, implying the existence of both types of algebraic solitons. See figure 6. In conclusion, we emphasize that the criterion for the existence of solitons (which depends crucially on the sign of κ) plays an important role in our analysis.

4.2. Two-soliton solution

As clarified by the analysis of the one-soliton solutions, both dark and bright solitons exist in our system. Therefore, the two-soliton solutions can be classified into three types, i.e., dark-dark solitons, dark-bright solitons and bright-bright solitons. Here, we focus our attention on the dark-dark solitons. Especially, we investigate the asymptotic behavior of the solution for large time. The two-soliton solution describing the interaction between a dark soliton and a bright soliton will be briefly discussed. For both cases, we assume that $\kappa > 0$.

4.2.1. Dark-dark solitons

The tau functions f_2 and g_2 representing the dark two-soliton solution are given by (3.1)-

(3.3) with $N = 2$ subjected to the conditions $\kappa > 0, a_1 > 0, a_2 > 0$. They read

$$f_2 = 1 + \frac{\kappa - ip_1}{p_1 + p_1^*} z_1 z_1^* + \frac{\kappa - ip_2}{p_2 + p_2^*} z_2 z_2^* + \frac{(\kappa - ip_1)(\kappa - ip_2)(p_1 - p_2)(p_1^* - p_2^*)}{(p_1 + p_1^*)(p_1 + p_2^*)(p_2 + p_1^*)(p_2 + p_2^*)} z_1 z_2 z_1^* z_2^*, \quad (4.28a)$$

$$g_2 = 1 - \frac{\kappa + ip_1^* p_1}{p_1 + p_1^* p_1^*} z_1 z_1^* - \frac{\kappa + ip_2^* p_2}{p_2 + p_2^* p_2^*} z_2 z_2^* + \frac{(\kappa + ip_1^*)(\kappa + ip_2^*)(p_1 - p_2)(p_1^* - p_2^*) p_1 p_2}{(p_1 + p_1^*)(p_1 + p_2^*)(p_2 + p_1^*)(p_2 + p_2^*) p_1^* p_2^*} z_1 z_2 z_1^* z_2^*. \quad (4.28b)$$

To investigate the interaction process of two solitons, we first order the magnitude of the velocity of each soliton in the (x, t) coordinate system as $c_1 > c_2 > 0$. Invoking the definition (4.2a) of the velocity of the solitons, this can be established by imposing the condition $|p_1| < |p_2|$ on the amplitude parameters. Now, we take the limit $t \rightarrow -\infty$ with θ_1 being fixed. Since in this limit $|z_1| = \text{finite}$ and $|z_2| \rightarrow \infty$, the leading-order asymptotics of f_2 and g_2 are found to be as

$$f_2 \sim \frac{\kappa - ip_2}{p_2 + p_2^*} z_2 z_2^* \left\{ 1 + \frac{(\kappa - ip_1)(p_1 - p_2)(p_1^* - p_2^*)}{(p_1 + p_1^*)(p_1 + p_2^*)(p_2 + p_1^*)} z_1 z_1^* \right\}, \quad (4.29a)$$

$$g_2 \sim -\frac{\kappa + ip_2^* p_2}{p_2 + p_2^* p_2^*} z_2 z_2^* \left\{ 1 - \frac{(\kappa + ip_1^*)(p_1 - p_2)(p_1^* - p_2^*) p_1}{(p_1 + p_1^*)(p_1 + p_2^*)(p_2 + p_1^*) p_1^*} z_1 z_1^* \right\}. \quad (4.29b)$$

The asymptotic form of the two-dark soliton solution follows from (2.1) upon substituting (4.29) into it, giving rise to

$$u_2 \sim \rho \exp \left\{ i \left(\kappa x - \omega t + \phi_1^{(-)} \right) \right\} \frac{1 - \frac{\kappa + ip_1^* p_1}{p_1 + p_1^* p_1^*} z_1' z_1'^*}{1 + \frac{\kappa - ip_1}{p_1 + p_1^*} z_1' z_1'^*}, \quad (4.30a)$$

where

$$z_1' = z_1 \exp \left[-\ln \left(\frac{p_1 + p_2^*}{p_1 - p_2} \right) \right], \quad (4.30b)$$

$$\phi_1^{(-)} = \arg \left(\frac{\kappa + ip_2^* p_2}{\kappa - ip_2 p_2^*} \right) + \pi. \quad (4.30c)$$

Let $u_1(\theta_1)$ be the dark one-soliton solution (4.10). Then, the asymptotic form of u_2 can be written in terms of u_1 as

$$u_2 \sim \exp \left(i \phi_1^{(-)} \right) u_1(\theta_1 + \Delta \theta_1^{(-)}), \quad \Delta \theta_1^{(-)} = -\ln \left| \frac{p_1 + p_2^*}{p_1 - p_2} \right|. \quad (4.31)$$

Next, we take the limit $t \rightarrow +\infty$ with θ_1 being fixed. In this limit, $|z_1|$ =finite and $|z_2| \rightarrow 0$. Therefore, the tau functions f_2 and g_2 and the two-soliton solution u_2 behave like

$$f_2 \sim 1 + \frac{\kappa - ip_1}{p_1 + p_1^*} z_1 z_1^*, \quad g_2 \sim 1 - \frac{\kappa + ip_1^* p_1}{p_1 + p_1^* p_1^*} z_1 z_1^*, \quad (4.32)$$

$$u_2 \sim \rho e^{i(\kappa x - \omega t)} \frac{1 - \frac{\kappa + ip_1}{p_1 + p_1^*} \frac{p_1}{p_1^*} z_1 z_1^*}{1 + \frac{\kappa - ip_1}{p_1 + p_1^*} z_1 z_1^*}. \quad (4.33)$$

It follows from (4.33) that

$$u_2 \sim u_1(\theta_1 + \Delta\theta_1^{(+)}) , \quad \Delta\theta_1^{(+)} = 0. \quad (4.34)$$

The trajectory of the center position $x = x_c(t)$ of the j th soliton is described by the equation $\theta_j + \Delta\theta_j^{(\pm)} = 0$, or $x_c = -c_j t - (\theta_{j0} + \Delta\theta_j^{(\pm)})/a_j$. Since the soliton propagates to the left, the phase shift Δx_j of the j th soliton can be defined by the relation

$$\Delta x_j = x_c(-\infty) - x_c(+\infty) = \frac{1}{a_j} \left(\Delta\theta_j^{(+)} - \Delta\theta_j^{(-)} \right), \quad j = 1, 2. \quad (4.35)$$

We see from (4.31) and (4.34) that the fast soliton suffers a phase shift

$$\Delta x_1 = \frac{1}{a_1} \ln \left| \frac{p_1 + p_2^*}{p_1 - p_2} \right|. \quad (4.36)$$

In terms of the angular variable γ_1 and γ_2 defined by (4.6) and (4.7), this expression can be rewritten in the form

$$\Delta x_1 = \frac{1}{a_1} \ln \left| \frac{\sin \frac{1}{2} (\gamma_1 + \gamma_2)}{\sin \frac{1}{2} (\gamma_1 - \gamma_2)} \right|, \quad a_1 = \sqrt{\kappa^3 \rho^2 (1 + \kappa \rho^2)} \sin \gamma_1, \quad 0 < \gamma_1 < \pi. \quad (4.37)$$

We can perform the similar asymptotic analysis while keeping θ_2 fixed. Hence, we quote only the final results. As $t \rightarrow -\infty$, the expressions corresponding to (4.29) and (4.31) read respectively

$$f_2 \sim 1 + \frac{\kappa - ip_2}{p_2 + p_2^*} z_2 z_2^*, \quad g_2 \sim 1 - \frac{\kappa + ip_2^* p_2}{p_2 + p_2^* p_2^*} z_2 z_2^*, \quad (4.38)$$

$$u_2 \sim u_1(\theta_2 + \Delta\theta_2^{(-)}) , \quad \Delta\theta_2^{(-)} = 0. \quad (4.39)$$

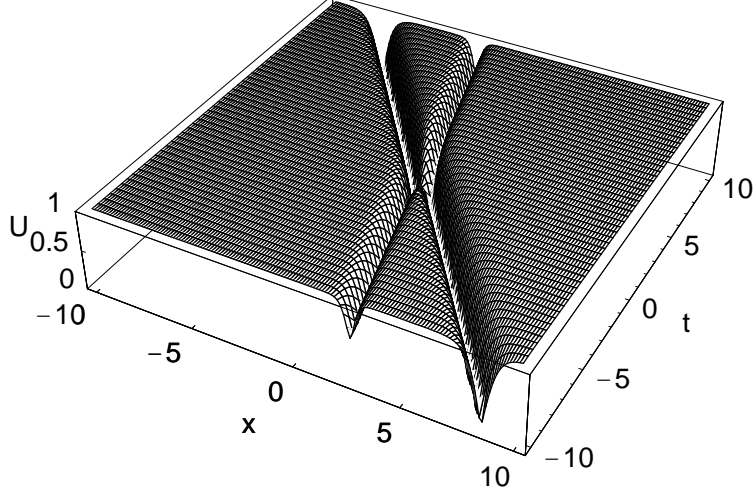


Figure 11. The interaction of two dark solitons.

As $t \rightarrow \infty$, on the other hand, they take the form

$$f_2 \sim \frac{\kappa - ip_1}{p_1 + p_1^*} z_1 z_1^* \left\{ 1 + \frac{(\kappa - ip_2)(p_1 - p_2)(p_1^* - p_2^*)}{(p_2 + p_2^*)(p_1 + p_2^*)(p_2 + p_1^*)} z_2 z_2^* \right\}, \quad (4.40a)$$

$$g_2 \sim -\frac{\kappa + ip_1}{p_1 + p_1^*} \frac{p_1}{p_1^*} z_1 z_1^* \left\{ 1 - \frac{(\kappa + ip_2^*)(p_1 - p_2)(p_1^* - p_2^*)}{(p_2 + p_2^*)(p_1 + p_2^*)(p_2 + p_1^*)} \frac{p_2}{p_2^*} z_2 z_2^* \right\}, \quad (4.40b)$$

$$u_2 \sim \exp(i\phi_2^{(+)}) u_1(\theta_2 + \Delta\theta_2^{(+)}), \quad (4.41a)$$

$$\Delta\theta_2^{(+)} = -\ln \left| \frac{p_2 + p_1^*}{p_2 - p_1} \right|, \quad \phi_2^{(+)} = \arg \left(\frac{\kappa + ip_1^*}{\kappa - ip_1} \frac{p_1}{p_1^*} \right) + \pi. \quad (4.41b)$$

The phase shift of the slow soliton follows from (4.35), (4.39) and (4.41), resulting in

$$\Delta x_2 = -\frac{1}{a_2} \ln \left| \frac{p_2 + p_1^*}{p_2 - p_1} \right|, \quad (4.42)$$

or equivalently in terms of the angular variables γ_1 and γ_2 , it reads

$$\Delta x_2 = -\frac{1}{a_2} \ln \left| \frac{\sin \frac{1}{2}(\gamma_2 + \gamma_1)}{\sin \frac{1}{2}(\gamma_2 - \gamma_1)} \right|, \quad a_2 = \sqrt{\kappa^3 \rho^2 (1 + \kappa \rho^2)} \sin \gamma_2, \quad 0 < \gamma_2 < \pi. \quad (4.43)$$

An inspection of the formulas (4.36) and (4.42) reveals that $\Delta x_1 > 0$ and $\Delta x_2 < 0$ under the setting $a_1 > 0$, $a_2 > 0$.

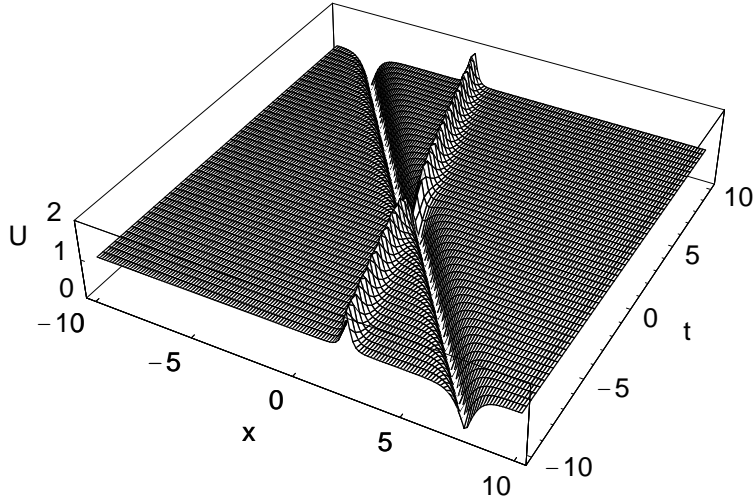


Figure 12. The interaction between a dark soliton and a bright soliton.

Figure 11 shows the interaction of two dark solitons with the parameters $\rho = 1, \kappa = 2, c_1 = 0.75(\gamma_1 = 0.90\pi), c_2 = 0.24(\gamma_2 = 0.80\pi)$ and $\zeta_{10} = \zeta_{20} = 0$ so that from (4.14), $A_{d1} = 1.0$ and $A_{d2} = 0.47$. It can be seen from figure 1 that the amplitude of each dark soliton is an increasing function of the velocity for the present choice of the parameters. Note, in this example, that the large soliton is a black soliton since its asymptotic amplitude is $A_{d1} = \rho = 1$. The phase shifts evaluated from the formulas (4.37) and (4.43) are given by $\Delta x_1 = 0.70$ and $\Delta x_2 = -0.36$, respectively. Figure 11 shows clearly a typical interaction process of solitons, i.e., as time goes, the large soliton gets close to the small soliton and overtakes it and after the collision, both solitons eventually separate each other without changing their profiles. The net effect of the collision is only the phase shift.

4.2.2. Dark-bright solitons

The two-soliton solution consisting of a dark soliton and a bright soliton is obtained by choosing the parameters such as $\kappa > 0, a_1 > 0$ and $a_2 < 0$, for example. The asymptotic analysis can be performed as well for this solution and hence the detail will be omitted.

Figure 12 depicts the interaction between a dark soliton and a bright soliton with the parameters $\rho = 1, \kappa = 2, c_1 = 0.75(\gamma_1 = 0.90\pi), c_2 = 0.24(\gamma_2 = 1.2\pi)$ and $\zeta_{10} =$

$\zeta_{20} = 0$, showing that the dark soliton propagates faster than the bright soliton. The asymptotic amplitudes of the dark and bright solitons are given respectively by $A_{d1} = 1.0$ and $A_{b2} = 0.92$ and hence the former is a black soliton. The figure clearly shows the solitonic behavior of the solution. The dark soliton suffers a positive phase shift whereas the bright soliton suffers a negative phase shift. The formulas Δx_1 for the dark soliton and Δx_2 for the bright soliton for the phase shifts are given respectively by

$$\Delta x_1 = -\frac{1}{a_1} \ln \left| \frac{\sin \frac{1}{2}(\gamma_1 + \gamma_2)}{\sin \frac{1}{2}(\gamma_1 - \gamma_2)} \right|, \quad a_1 = \sqrt{\kappa^3 \rho^2 (1 + \kappa \rho^2)} \sin \gamma_1, \quad 0 < \gamma_1 < \pi, \quad (4.44a)$$

$$\Delta x_2 = -\frac{1}{a_2} \ln \left| \frac{\sin \frac{1}{2}(\gamma_2 + \gamma_1)}{\sin \frac{1}{2}(\gamma_2 - \gamma_1)} \right|, \quad a_2 = \sqrt{\kappa^3 \rho^2 (1 + \kappa \rho^2)} \sin \gamma_2, \quad \pi < \gamma_2 < 2\pi. \quad (4.44b)$$

As in the case of the dark-dark solitons, one can see that $\Delta x_1 > 0$ and $\Delta x_2 < 0$. In the present example, $\Delta x_1 = 0.70$ and $\Delta x_2 = -0.36$.

4.3. Dark N -soliton solution

The preceding analysis reveals that the asymptotic form of the N -soliton solution will be represented by a superposition of n dark solitons and $N - n$ bright solitons where n is an arbitrary nonnegative integer in the interval $0 \leq n \leq N$. The derivation of the large time asymptotic for the general N -soliton solution can be done following the similar procedure to that used for the two-soliton case. Hence, we outline the result. We address the dark soliton solutions satisfying the conditions $\kappa > 0$ and $a_j > 0$ ($j = 1, 2, \dots, N$). The analysis for the bright soliton solutions as well as an arbitrary combination of dark and bright solitons can be carried out in exactly the same way.

To begin with, we order the magnitude of the velocity of each soliton as $c_1 > c_2 > \dots > c_N > 0$. We take the limit $t \rightarrow -\infty$ with θ_n being finite. Since in this limit, $|z_j| \rightarrow 0$ for $j < n$ and $|z_j| \rightarrow \infty$ for $n < j$, we find that the leading-order asymptotic of the tau function $f = f_N$ from (3.1) with (3.2) can be written in the form

$$f_N \sim |(c_{jk})_{n+1 \leq j, k \leq N}| \prod_{j=n+1}^N (z_j z_j^*) \left(1 + \frac{|(c_{jk})_{n \leq j, k \leq N}|}{|(c_{jk})_{n+1 \leq j, k \leq N}|} z_n z_n^* \right). \quad (4.45a)$$

Here, (c_{jk}) is a matrix of Cauchy type given by

$$c_{jk} = \frac{\kappa - ip_j}{p_j + p_k^*}, \quad 1 \leq j, k \leq N. \quad (4.45b)$$

Referring to the well-known Cauchy's formula, the determinant of the matrix (c_{jk}) is evaluated as

$$|(c_{jk})_{m \leq j, k \leq n}| = \prod_{j=m}^n (\kappa - ip_j) \frac{\prod_{m \leq j < k \leq n} (p_j - p_k)(p_j^* - p_k^*)}{\prod_{m \leq j, k \leq n} (p_j + p_k^*)}, \quad 1 \leq m < n \leq N. \quad (4.45c)$$

If we use (4.45c), we have

$$\frac{|(c_{jk})_{n \leq j, k \leq N}|}{|(c_{jk})_{n+1 \leq j, k \leq N}|} = \frac{\kappa - ip_n}{p_n + p_n^*} \exp \left[- \sum_{j=n+1}^N \ln \left(\frac{p_n + p_j^*}{p_n - p_j} \right) - \sum_{j=n+1}^N \ln \left(\frac{p_n^* + p_j}{p_n^* - p_j^*} \right) \right]. \quad (4.46)$$

Substitution of (4.46) into (4.45) now gives

$$f_N \sim |(c_{jk})_{n+1 \leq j, k \leq N}| \prod_{j=n+1}^N (z_j z_j^*) \left(1 + \frac{\kappa - ip_n}{p_n + p_n^*} z_n' z_n'^* \right), \quad (4.47a)$$

where

$$z_n' = z_n \exp \left[- \sum_{j=n+1}^N \ln \left(\frac{p_n + p_j^*}{p_n - p_j} \right) \right]. \quad (4.47b)$$

The leading-order asymptotic of g_N in the limit of $t \rightarrow -\infty$ can be derived in the same way. It takes the form

$$g_N \sim |(c'_{jk})_{n+1 \leq j, k \leq N}| \prod_{j=n+1}^N (z_j z_j^*) \left(1 - \frac{\kappa + ip_n^*}{p_n + p_n^*} \frac{p_n}{p_n^*} z_n' z_n'^* \right), \quad (4.48a)$$

where

$$c'_{jk} = - \frac{\kappa - ip_j}{p_j + p_k^*} \frac{p_j}{p_k^*}, \quad 1 \leq j, k \leq N. \quad (4.48b)$$

The asymptotic form of the dark N -soliton solution follows from (2.1), (4.47) and (4.48).

It reads

$$u_N \sim \rho \exp \{ i (\kappa x - \omega t + \phi_n^{(-)}) \} \frac{1 - \frac{\kappa + ip_n^*}{p_n + p_n^*} \frac{p_n}{p_n^*} z_n' z_n'^*}{1 + \frac{\kappa - ip_n}{p_n + p_n^*} z_n' z_n'^*}, \quad (4.49a)$$

with

$$\phi_n^{(-)} = \arg \left[\prod_{j=n+1}^N \left(\frac{\kappa + ip_j^*}{\kappa - ip_j} \frac{p_j}{p_j^*} \right) \right] + (N - n)\pi. \quad (4.49b)$$

This expression can be rewritten in terms of the one-soliton solution as

$$u_N \sim \exp (i\phi_n^{(-)}) u_1(\theta_n + \Delta\theta_n^{(-)}), \quad (4.50a)$$

with

$$\Delta\theta_n^{(-)} = - \sum_{j=n+1}^N \ln \left| \frac{p_n + p_j^*}{p_n - p_j} \right|. \quad (4.50b)$$

By a similar asymptotic analysis, we can derive the asymptotic form of u_N in the limit of $t \rightarrow +\infty$. We find that

$$u_N \sim \exp(i\phi_n^{(+)}) u_1(\theta_n + \Delta\theta_n^{(+)}, \quad (4.51a)$$

with

$$\Delta\theta_n^{(+)} = - \sum_{j=1}^{n-1} \ln \left| \frac{p_n + p_j^*}{p_n - p_j} \right|, \quad (4.51b)$$

$$\phi_n^{(+)} = \arg \left[\prod_{j=1}^{n-1} \left(\frac{\kappa + ip_j^*}{\kappa - ip_j} \frac{p_j}{p_j^*} \right) \right] + (n-1)\pi. \quad (4.51c)$$

We see from (4.50) and (4.51) that in the rest frame of reference, the asymptotic form of the dark N -soliton solution can be represented by a superposition of N independent dark one-soliton solutions, the only difference being the phase shifts of each soliton caused by the collisions. It follows from (4.50b) and (4.51b) that the formula for the total phase shift of the n th soliton is given by

$$\Delta x_n = \frac{1}{a_n} \left(\sum_{j=n+1}^N \ln \left| \frac{p_n + p_j^*}{p_n - p_j} \right| - \sum_{j=1}^{n-1} \ln \left| \frac{p_n + p_j^*}{p_n - p_j} \right| \right), \quad n = 1, 2, \dots, N. \quad (4.52)$$

As in the two-soliton case, we can rewrite the above formula in terms of the variables γ_j defined by (4.6) and (4.7). Explicitly,

$$\Delta x_n = \frac{1}{a_n} \left(\sum_{j=n+1}^N \ln \left| \frac{\sin \frac{1}{2}(\gamma_n + \gamma_j)}{\sin \frac{1}{2}(\gamma_n - \gamma_j)} \right| - \sum_{j=1}^{n-1} \ln \left| \frac{\sin \frac{1}{2}(\gamma_n + \gamma_j)}{\sin \frac{1}{2}(\gamma_n - \gamma_j)} \right| \right),$$

$$a_n = \sqrt{\kappa^3 \rho^2 (1 + \kappa \rho^2)} \sin \gamma_n, \quad 0 < \gamma_n < \pi, \quad n = 1, 2, \dots, N. \quad (4.53)$$

The formulas (4.52) and (4.53) reduce to (4.36), (4.37), (4.42) and (4.43) for the special case of $N = 2$. They clearly show that each soliton has pairwise interactions with other solitons, i.e., there are no many-particle collisions among solitons. This feature is common to that of the bright N -soliton solution considered in I.

5. Concluding remarks

In this paper, the system of bilinear equations reduced from the FL equation has been derived and used to construct the dark N -soliton solution. The corresponding N -soliton solution derived in [7] using the Bäcklund transformation follows from our solution (2.1) with (3.1) and (3.2) if one introduces the angular variables γ_j according to the relations (4.7). We have found that unlike the bright soliton solutions obtained in I, the complex amplitude parameters p_j are subjected to the constraints (3.2c) which have prevented the proof of the solution. To overcome this difficulty, we have employed a trilinear equation in place of one of the bilinear equations, in addition to an auxiliary variable τ in (3.2c). As a byproduct, this trilinear equation has led for the first time to a simple formula for the dark N -soliton solution of the derivative NLS equation on the background of a plane wave. Note that the dark soliton solutions on a constant background [13-19] stem simply from the above-mentioned solution in the zero limit of the wavenumber κ . However, this limiting procedure is found to be unable to perform for the dark N -soliton solution of the FL equation due to the singularity of the dispersion relation.

We have seen that the soliton solutions presented here exhibit several new features. Specifically, both the dark and bright solitons exist depending on the sign of the wavenumber κ and that of the real part of the complex amplitude parameter. Of particular interest is the existence of an algebraic dark soliton which appears only in the case of negative κ . Finally, the asymptotic analysis of the two- and general N -soliton solutions has clarified their structure and dynamics. In particular, the latter solution has been shown to include n dark solitons and $N - n$ bright solitons on nonzero background with n being an arbitrary nonnegative integer not exceeding N . The application of the results summarized above to nonlinear fiber optics will be an interesting issue to be studied in a future research work.

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References

- [1] Fokas A S 1995 On a class of physically important integrable equations *Physica D* **87** 145-150
- [2] Lenells J 2009 Exactly solvable model for nonlinear pulse propagation in optical fibers *Stud. Appl. Math.* **123** 215-232
- [3] Lenells J and Fokas A S 2009 On a novel integrable generalization of the nonlinear Schrödinger equation *Nonlinearity* **22** 11-27
- [4] Lenells J 2010 Dressing for a novel integrable generalization of the nonlinear Schrödinger equation *J. Nonlinear Sci.* **20** 709-722
- [5] Kundu A 2010 Two-fold integrable hierarchy of nonholonomic deformation of the derivative nonlinear Schrödinger and the Lenells-Fokas equation *J. Math. Phys.* **51** 022901
- [6] Matsuno Y 2011 A direct method of solution for the Fokas-Lenells derivative nonlinear Schrödinger equation: I. Bright soliton solutions *J. Phys. A: Math. Theor.* **45** 235202
- [7] Vekslerchik V E 2011 Lattice representation and dark solitons of the Fokas-Lenells equation *Nonlinearity* **24** 1165-1175
- [8] Hirota R 2004 *The Direct Method in Soliton Theory* (New York: Cambridge)
- [9] Matsuno Y 1984 *Bilinear Transformation Method* (New York: Academic)
- [10] Vein R and Dale P 1999 *Determinants and Their Applications in Mathematical Physics* (New York: Springer)
- [11] Matsuno Y 2011 The N -soliton solution of a two-component modified nonlinear Schrödinger equation *Phys. Lett. A* **375** 3090-3094
- [12] Matsuno Y 2011 The bright N -soliton solution of a multi-component modified nonlinear Schrödinger equation *J. Phys. A: Mat. Theor.* **44** 495202

- [13] Kawata T and Inoue H 1978 Exact solutions of derivative nonlinear Schrödinger equation under the nonvanishing conditions *J. Phys. Soc. Jpn.* **44** 1968-1976
- [14] Kawata T, Kobayashi N and Inoue H 1979 Soliton solutions of the derivative nonlinear Schrödinger equation *J. Phys. Soc. Jpn.* **46** 1008-1015
- [15] Chen X J, Yang J and Lam W K 2006 N -soliton solution for the derivative nonlinear Schrödinger equation with nonvanishing boundary conditions *J. Phys. A: Math. Gen.* **39** 3263-3274
- [16] Laskin V M 2007 N -soliton solutions and perturbation theory for the derivative nonlinear Schrödinger equation with nonvanishing boundary condition *J. Phys. A: Math. Theor.* **40** 6119-6132
- [17] Steudel H 2003 The hierarchy of multi-soliton solutions of the derivative nonlinear Schrödinger equation *J. Phys. A: Math. Gen.* **36** 1931-1946
- [18] Xu S, He J and Wang L 2011 The Darboux transformation of the derivative nonlinear Schrödinger equation *J. Phys. A: Math. Theor.* **44** 305203
- [19] Li M, Tian B, Liu W J, Zhang H Q and Wang P 2010 Dark and antidark solitons in the modified nonlinear Schrödinger equation accounting for the self-steepening effect *Phys. Rev. E* **81** 046606
- [20] Mjølhus E 1976 On the modulational instability of hydromagnetic waves parallel to the magnetic field *J. Plasma Phys.* **16** 321-334
- [21] Ichikawa Y H, Konno K, Wadati M and Sanuki H 1980 Spiky soliton in circular polarized Alfvén wave *J. Phys. Soc. Jpn.* **48** 279-286
- [22] Wright III O C 2009 Some homoclinic connections of a novel integrable generalized nonlinear Schrödinger equation *Nonlinearity* **22** 2633-2643
- [23] Lü X and Tian B 2012 Novel behavior and properties for the nonlinear pulse propagation in optical fibers *Europhys. Lett.* **97** 10005

- [24] Mio K, Ogino T, Minami K and Takeda S 1976 Modified nonlinear Schrödinger equation for Alfvén waves propagating along the magnetic field in cold plasmas *J. Phys. Soc. Jpn.* **41** 265-271
- [25] Mijøllhus E 1978 A note on the modulational instability of long Alfvén waves parallel to the magnetic field *J. Plasma Phys.* **19** 437-447